JEE-Main-27-07-2022-Shift-2 (Memory Based)

MATHEMATICS

Question: Let $A=\begin{bmatrix}4 & -2\\ \alpha & \beta\end{bmatrix}$. If $A^2+\gamma A+18I=0$, then $\det(A)$ equals:

Options:

(a) -18

(b) 18

(c) -50

(d) 50

Answer: (b)

Solution:

Characteristic equation of matrix:

$\begin{bmatrix}4-\lambda & -2\\ \alpha & \beta-\lambda\end{bmatrix}=0$

$\Rightarrow 4\beta+\lambda^2-(\beta+4)\lambda+2\alpha=0$

$\therefore A^2-(\beta+4)A+2\alpha I=0$

$\Rightarrow \gamma=0-\beta+4\ \&\ 2\alpha+4\beta=18$

$\det(A)=4\beta+2\alpha=18$

Question: The area of region enclosed by $y\le 4x^2, x^2\le 9y, y\le 4$ is equal to:

Options:

(a) $\frac{40}{3}$

(b) $\frac{56}{3}$

(c) $\frac{112}{3}$

(d) $\frac{80}{3}$

Answer: (d)

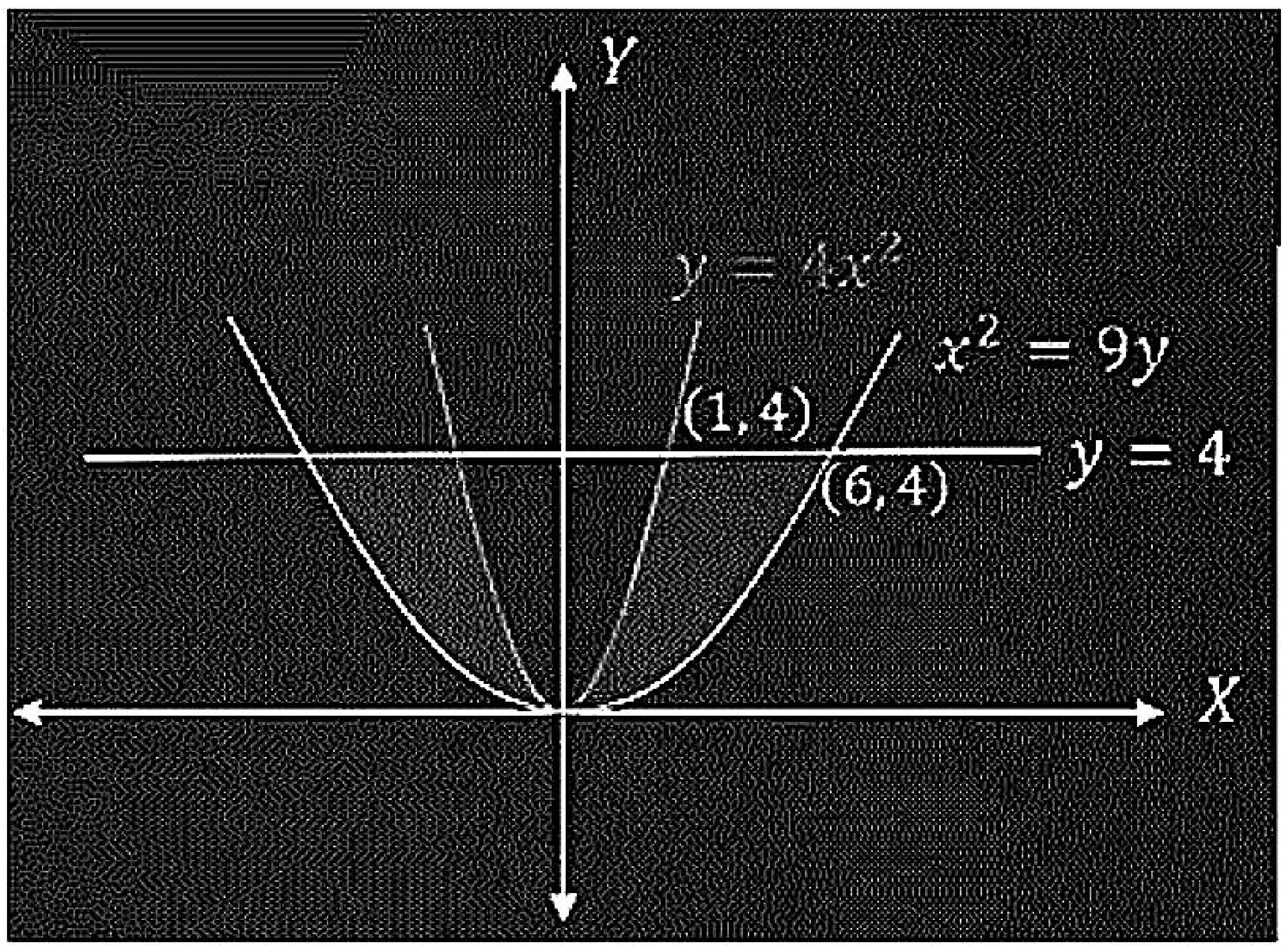
Solution:

Required Area $=2\int_0^4\left(3\sqrt{y}-\frac{\sqrt{y}}{2}\right)dy$

$=2\cdot\frac{5}{2}\int_0^4\sqrt{y}dy$

$=5\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^4$

$= \frac{10}{3}(4)^{\frac{3}{2}} = \frac{80}{3}$



Question: If the length of the latus rectum of a parabola whose focus is (a, a) and tangent at its vertex is $x + y = a$, is 16. Then $|a|$ is equal to:

Options:

(a) $2\sqrt{3}$

(b) $2\sqrt{2}$

(c) $4\sqrt{2}$

(d) 4

Answer: (c)

Solution:

Length of perpendicular from focus to tangent at vertex:

$l = \left|\frac{a}{\sqrt{2}}\right|$

So length of latus rectum will be, $4l = 16$

$\Rightarrow 2\sqrt{2}|a| = 16$

$\Rightarrow |a| = 4\sqrt{2}$

Question: Let $f(x) = \frac{\left(729p(1+x)^{\frac{1}{7}}\right) - 3}{\left(729(1+qx)^{\frac{1}{3}}\right) - 9}$, and $f(x)$ is continuous at $x = 0$, then:

Options:

(a) $21qf(0) - p = 0$

(b) $21q^2 f(0)-p^3=0$

(c) $21p^2 f(0)-q^3=0$

(d) $p^2 f(0)-7q^2=0$

Answer: (a)

Solution:

$\lim_{x\to 0} f(x)$ exists if numerator of $f(x)$ is zero at $x=0$.

Clearly, $p=3$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{3\left[(x+1)^{\frac{1}{7}}-1\right]}{9\left[(1+qx)^{\frac{1}{3}}-1\right]}$$

$$=\frac{1}{3}\left(\frac{\frac{1}{7}}{\frac{q}{3}}\right)=\frac{1}{7q}=f(0)$$

So, $21qf(0)=3=p$

$\Rightarrow 21qf(0)-p=0$

Question: Let $f(x)=\min\{[x],[x-1],[x-2],...,[x-10]\}$ where $[\ ]$ denotes greatest integer function. Then $\int_0^{10}\left(f(x)+|f(x)|+f^2(x)\right)dx$ is equal to:

Options:

(a) 55

(b) 385

(c) 5050

(d) 270

Answer: (b)

Solution:

Clearly $f(x)=[x-10]$

Here $f(x)\le 0\ \forall\ x\in(0,10)$

So, $\int_0^{10}\left(f(x)+|f(x)|\right)dx=0$

Now, $\int_0^{10} f^2(x)\,dx=\int_0^{10}\left([x]-10\right)^2 dx$

$$=\int_0^1 100\,dx+\int_1^2 81\,dx+\int_2^3 64\,dx+...+\int_9^{10} 1\,dx$$

$=\left(1^2+2^2+3^2+....+10^2\right)$

$=\dfrac{10\times 11\times 21}{6}$

$=385$

Question: The value of $\int_0^2\left(\left|2x^3-3x\right|+\left[x-\frac{1}{2}\right]\right)dx$, where $[.]$ is greatest integer function is:

Options:

(a) $\frac{7}{6}$

(b) $\frac{19}{12}$

(c) $\frac{17}{4}$

(d) $\frac{3}{2}$

Answer: (c)

Solution:

Given, $\int_0^2\left(\left|2x^3-3x\right|+\left[x-\frac{1}{2}\right]\right)dx$

$\int_0^2\left|2x^3-3x\right|dx+\int_0^2\left[x-\frac{1}{2}\right]dx$

$=\int_0^{\sqrt{\frac{3}{2}}}\left(3x-2x^3\right)dx+\int_{\sqrt{\frac{3}{2}}}^{2}\left(2x^3-3x\right)dx+\int_0^{\frac{1}{2}}\left[x-\frac{1}{2}\right]dx+\int_{\frac{1}{2}}^{\frac{3}{2}}\left[x-\frac{1}{2}\right]dx+\int_{\frac{3}{2}}^{2}\left[x-\frac{1}{2}\right]dx$

$=\left[\frac{3x^2-x^4}{2}\right]_0^{\sqrt{\frac{3}{2}}}+\left[\frac{x^4-3x^2}{2}\right]_{\sqrt{\frac{3}{2}}}^{2}+\left(-\frac{1}{2}\right)+0+\left(\frac{1}{2}\right)$

$=\frac{9}{8}+2+\frac{9}{8}$

$=\frac{17}{4}$

Question: If the line of intersection of the planes $ax+by=3$ and $ax+by+cz=0$ makes an angle 30° with the plane $y-z+2=0$, then the direction cosines of line are:

Options:

(a) $\frac{1}{\sqrt{2}},0,\frac{1}{2}$

(b) $\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0$

(c) $\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}},0$

(d) $\frac{1}{2},-\frac{\sqrt{3}}{2},0$

Answer: (b)

Solution:

Direction ratios of line of intersection $(b,-a,0)$

As angle between this line and $y-z+2=0$ is $30°$

$\therefore\ \sin\theta=\left|\frac{a}{\sqrt{a^2+b^2}\cdot\sqrt{2}}\right|$

$\Rightarrow a^2=b^2$

$\therefore$ Possible combination is $\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0$

Question: If $A=\begin{bmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \alpha+\gamma & \alpha+\beta\end{bmatrix}$ and $\frac{\left|adj\left(adj\left(adj\left(adj(A)\right)\right)\right)\right|}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}}=2^{32}\cdot 3^{16}$, where α,β,γ are distinct natural number, then number of triplets of (α,β,γ) is _____.

Answer: 55.00

Solution:

$$A=\begin{bmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \alpha+\gamma & \alpha+\beta\end{bmatrix}$$

$R_3\to R_3+R_1$

$$\Rightarrow |A|=(\alpha+\beta+\gamma)\begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ 1 & 1 & 1\end{vmatrix}$$

$=(\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

$\because \left|adj\left(adj\left(adj\left(adj(A)\right)\right)\right)\right|=|A|^{(2)^4}=|A|^{16}$

$|A|=(\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

Clearly $(\alpha+\beta+\gamma)^{16}=2^{32}\cdot 3^{16}$

$\Rightarrow(\alpha+\beta+\gamma)=12$

Number of positive integral solutions $={}^{11}C_2=55$

Question: $\frac{(2^3-1^3)}{(1\times 7)}+\frac{\{(4^3-3^3)+(2^3-1^3)\}}{(2\times 11)}+\frac{\{(6^3-5^3)+(4^3-3^3)+(2^3-1^3)\}}{(3\times 15)}+\ldots$ upto 15 terms

Answer: 120.00

Solution:

$$\frac{2^3-1^3}{1\times 7}+\frac{4^3-3^3+2^3-1^3}{2\times 11}+\ldots.$$

$$=1+2+3+\ldots.$$

$$=\left(\frac{15\times 16}{2}\right)$$

$$=120$$

Question: Domain of $f(x)=\sin^{-1}\left[2x^2-3\right]+\log_2\left(\log_{\frac{1}{2}}\left(x^2-5x+5\right)\right)$

Answer: $1,\frac{5-\sqrt{5}}{2}$

Solution:

$$-1\le\left[2x^2-3\right]\le 1$$

$$-1\le\left(2x^2-3\right)<2$$

$$2\le 2x^2<5$$

$$1\le x^2<\frac{5}{2} \quad \ldots.(1)$$

$$\log_{\frac{1}{2}}\left(x^2-5x+5\right)>0$$

$$0<x^2-5x+5<1$$

$$\Rightarrow x^2-5x+5=0 \text{ and } x^2-5x+4<0$$

$$x\in\left(-\infty,\frac{5-\sqrt{5}}{2}\right)\cup\left(\frac{5+\sqrt{5}}{2},\infty\right) \quad \ldots.(2)$$

and $x\in(1,4)$

Taking intersection of (1) and (2)

$$x\in\left(1,\frac{5-\sqrt{5}}{2}\right)$$

Question: Let n^{th} term of any sequence is given by $T_n=\frac{-1^3+2^3-3^3+4^3+\ldots+(2n)^3}{n(4n+3)}$, then $\sum_{n=1}^{15}T_n$ is equal to ____.

Answer: 120.00

Solution:

$$T_n = \frac{2\left[2^3 + 4^3 + ... + \left(2n^3\right)\right] - \left[1^3 + 2^3 + 3^3 + ... + \left(2n\right)^3\right]}{n\left(4n+3\right)}$$

$$T_n = \frac{16\left(\frac{n\left(n+1\right)}{2}\right)^2 - \left(\frac{2n\left(2n+1\right)}{2}\right)^2}{n\left(4n+2\right)}$$

$$= \frac{n^2\left(4n+3\right)}{n\left(4n+3\right)} = n$$

$$\therefore \sum_{n=1}^{15} T_n = \frac{15\times 16}{2} = 120$$