SECTION – A

1. If the Boolean expression $(p \wedge q) \circledast (p \otimes q)$ is a tautology, then $\circledast$ and $\otimes$ are respectively given by :

(1) $\wedge, \rightarrow$

(2) $\rightarrow, \rightarrow$

(3) $\vee, \rightarrow$

(4) $\wedge, \vee$

Ans. (2)

Sol. $(p \wedge q) \rightarrow (p \rightarrow q)$

$(p \wedge q) \rightarrow (\sim p \vee q)$

$(\sim p \vee \sim q) \vee (\sim p \vee q)$

$\sim p \vee (\sim q \vee q) \Rightarrow$ Tautology

$\Rightarrow \circledast \Rightarrow \rightarrow$

$\otimes \Rightarrow \rightarrow$

2. Let the tangent to the circle $x^2 + y^2 = 25$ at the point R(3,4) meet x-axis and y-axis at points P and Q, respectively. If r is the radius of the circle passing through the origin O and having centre at the incentre of the triangle OPQ, then r^2 is equal to :

(1) $\frac{625}{72}$

(2) $\frac{585}{66}$

(3) $\frac{125}{72}$

(4) $\frac{529}{64}$

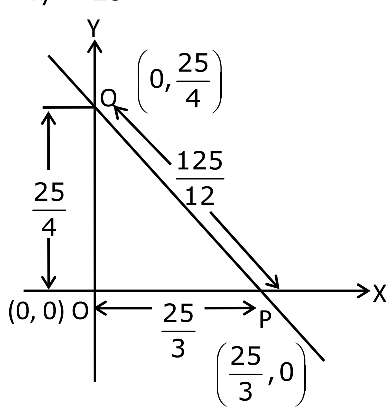
Ans. (1)

Sol. Given equation of circle

$x^2 + y^2 = 25$

$\therefore$ Tangent equation at (3, 4)

T : $3x + 4y = 25$



Incentre of ΔOPQ.

$$I = \left(\frac{\frac{25}{4}\times\frac{25}{3}}{\frac{25}{3}+\frac{25}{4}+\frac{125}{12}}, \frac{\frac{25}{3}\times\frac{25}{4}}{\frac{25}{3}+\frac{25}{4}+\frac{125}{12}}\right).$$

$\therefore \quad I = \left(\frac{625}{75+100+125}, \frac{625}{75+100+125}\right) = \left(\frac{25}{12}, \frac{25}{12}\right)$

$\because \quad$ Distance from origin to incentre is r.

$\therefore \quad r^2 = \left(\frac{25}{12}\right)^2 + \left(\frac{25}{12}\right)^2 = \frac{625}{72}$

Therefore, the correct answer is (1)

3. Let a computer program generate only the digits 0 and 1 to form a string of binary numbers with probability of occurrence of 0 at even places be $\frac{1}{2}$ and probability of occurrence of 0 at the odd place be $\frac{1}{3}$. Then the probability that '10' is followed by '01' is equal to :

(1) $\frac{1}{6}$

(2) $\frac{1}{18}$

(3) $\frac{1}{9}$

(4) $\frac{1}{3}$

Ans. (3)

Sol. P(0 at even place) = $\frac{1}{2}$, P(0 at odd place) = $\frac{1}{3}$

P(1 at even place) = $\frac{1}{2}$, P(1 at odd place) = $\frac{2}{3}$

P(10 is followed by 01)

$= \left(\frac{2}{3}\times\frac{1}{2}\times\frac{1}{3}\times\frac{1}{2}\right)+\left(\frac{1}{2}\times\frac{1}{3}\times\frac{1}{2}\times\frac{2}{3}\right)$

$= \frac{1}{18}+\frac{1}{18}$

$= \frac{1}{9}$

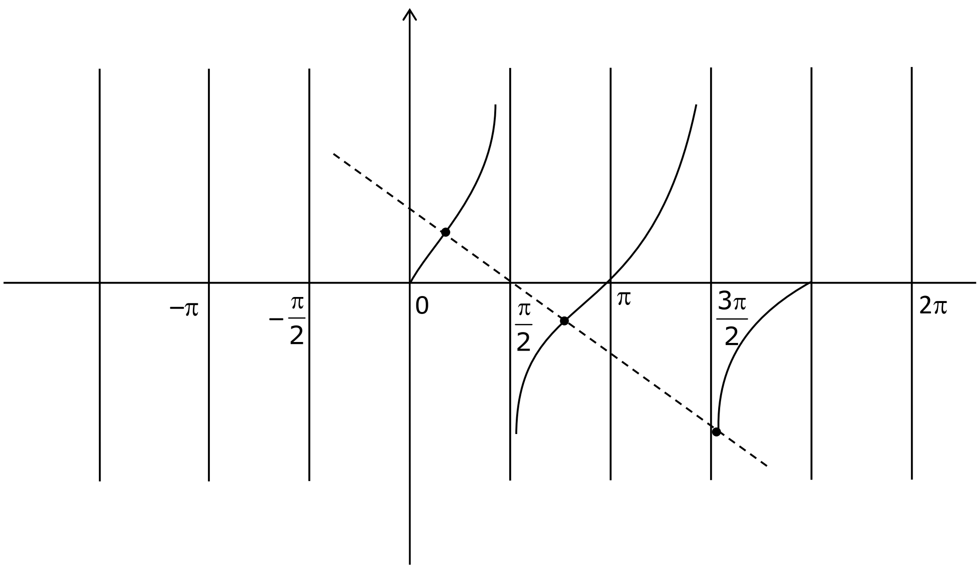
4. The number of solutions of the equation $x + 2\tan x = \frac{\pi}{2}$ in the interval $[0, 2\pi]$ is :

(1) 5 (2) 2

(3) 4 (4) 3

Ans. **(4)**

Sol.



$x + 2\tan x = \frac{\pi}{2}$ in $[0, 2\pi]$

$2\tan x = \frac{\pi}{2} - x$

$2\tan x = \frac{\pi}{2} - x$

$\tan x = \frac{\pi}{4} - \frac{x}{2}$

$y = \tan x$ and $y = \frac{-x}{2} + \frac{\pi}{4}$

3 intersection points

$\therefore$ 3 solutions

option (4)

5. If the equation of plane passing through the mirror image of a point (2, 3, 1) with respect to line $\frac{x+1}{2} = \frac{y-3}{1} = \frac{z+2}{-1}$ and containing the line $\frac{x-2}{3} = \frac{1-y}{2} = \frac{z+1}{1}$ is $\alpha x + \beta y + \gamma z = 24$, then $\alpha + \beta + \gamma$ is equal to :

(1) 21

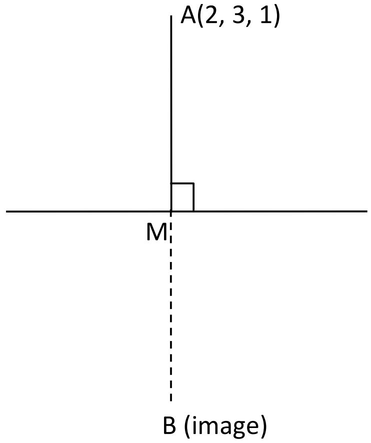
(2) 19

(3) 18

(4) 20

Ans. **(2)**

Sol.



$$\frac{x+1}{2} = \frac{y-3}{1} = \frac{z+2}{-1} \quad ...(L_1)$$

Let point M is $(2\lambda - 1, \lambda + 3, -\lambda - 2)$

D.R.'s of AM line are $2\lambda - 1 - 2,\quad \lambda + 3 - 3,\quad -\lambda - 2 - 1$

$2\lambda - 3,\quad \lambda,\quad -\lambda - 3$

AM $\perp$ line L_1

$\therefore\ 2(2\lambda - 3) + 1(\lambda) - 1(-\lambda - 3) = 0$

$6\lambda = 3, \lambda = \frac{1}{2}\ \therefore\ M \equiv \left(0, \frac{7}{2}, \frac{-5}{2}\right)$

M is mid-point of A & B

$M = \frac{A+B}{2}$

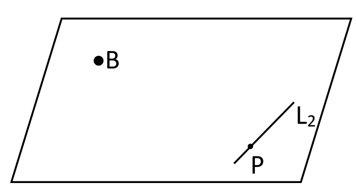
$B = 2M - A$

$B \equiv (-2, 4, -6)$

Now we have to find equation of plane passing through B (–2, 4, –6) & also containing the line

$\frac{x-2}{3} = \frac{1-y}{2} = \frac{z+1}{1} \quad ...(1)$

$\frac{x-2}{3} = \frac{y-1}{-2} = \frac{z+1}{1}$



Point P on line is (2, 1, –1)

$\vec{b}_2$ of line L_2 is 3, –2, 1

$\vec{n} \parallel (\vec{b}_2 \times \vec{PB})$

$\vec{b}_2 = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{PB} = -4\hat{i} + 3\hat{j} - 5\hat{k}$

$\vec{n} = 7\hat{i} + 11\hat{j} + \hat{k}$

$\therefore$ equation of plane is $\vec{r}\,.\,\vec{n} = \vec{a}\,.\,\vec{n}$

$\vec{r}\,.(7\hat{i} + 11\hat{j} + \hat{k}) = (-2\hat{i} + 4\hat{j} - 6\hat{k}).(7\hat{i} + 11\hat{j} + \hat{k})$

$7x + 11y + z = -14 + 44 - 6$

$7x + 11y + z = 24$

$\therefore \alpha = 7$

$\beta = 11$

$\gamma = 1$

$\therefore \alpha + \beta + \gamma = 19$

option (2)

6. Consider the function $f : R \to R$ defined by $f(x) = \begin{cases} \left(2 - \sin\left(\frac{1}{x}\right)\right)|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then f is :

(1) monotonic on $(0, \infty)$ only

(2) Not monotonic on $(-\infty, 0)$ and $(0, \infty)$

(3) monotonic on $(-\infty, 0)$ only

(4) monotonic on $(-\infty, 0) \cup (0, \infty)$

Ans. **(2)**

Sol.

$$f(x) = \begin{cases} -\left(2 - \sin\frac{1}{x}\right)x, & x < 0 \\ 0, & x = 0 \\ \left(2 - \sin\frac{1}{x}\right)x, & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -x\left(-\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) - \left(2 - \sin\frac{1}{x}\right), & x < 0 \\ x\left(-\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \left(2 - \sin\frac{1}{x}\right), & x > 0 \end{cases}$$

$$\begin{cases} -\frac{1}{x}\cos\frac{1}{x} + \sin\frac{1}{x} - 2, & x < 0 \\ \frac{1}{x}\cos\frac{1}{x} - \sin\frac{1}{x} + 2, & x > 0 \end{cases}$$

7. Let O be the origin. Let $\overrightarrow{OP} = x\hat{i} + y\hat{j} - \hat{k}$ and $\overrightarrow{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$, x, y $\in$ R, x > 0, be such that $|\overrightarrow{PQ}| = \sqrt{20}$ and the vector $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$. If $\overrightarrow{OR} = 3\hat{i} + z\hat{j} - 7\hat{k}$, z $\in$ R, is coplanar with $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the value of $x^2+y^2+z^2$ is equal to :

(1) 2 (2) 9

(3) 1 (4) 7

Ans. **(2)**

Sol. $\overrightarrow{OP} = x\hat{i} + y\hat{j} - \hat{k}$ $\overrightarrow{OP} \perp \overrightarrow{OQ}$

$\overrightarrow{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$

$\overrightarrow{PQ} = (-1-x)\hat{i} + (2-y)\hat{j} + (3x+1)\hat{k}$

$|\overrightarrow{PQ}| = \sqrt{(-1-x)^2 + (2-y)^2 + (3x+1)^2}$

$\sqrt{20} = \sqrt{(-1-x)^2 + (2-y)^2 + (3x+1)^2}$

$20 = 1 + x^2 + 2x + 4 + y^2 - 4y + 9x^2 + 1 + 6x$

$20 = 10x^2 + y^2 + 8x + 6 - 4y$

$20 = 10x^2 + 4x^2 + 8x + 6 - 8x$

$14 = 14x^2 \Rightarrow \boxed{x^2 = 1}$

$\therefore\ y^2 = 4x^2 \Rightarrow \boxed{y^2 = 4}$

x = 1 as x > 0 and y = 2

$\therefore \overrightarrow{OP}.\ \overrightarrow{OQ} = 0$

$-x + 2y - 3x = 0$

$4x = 2y$

$\boxed{y = 2x}$

$$\therefore \begin{vmatrix} x & y & -1 \\ -1 & 2 & 3x \\ 3 & z & -7 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & 3 \\ 3 & z & -7 \end{vmatrix}$$

$1(-14 - 3z) - 2(7 - 9) - 1(-z - 6)$

$-14 - 3z + 4 + z + 6 = 0$

$2z = -4$ $\boxed{z = -2}$

$x^2 + y^2 + z^2 = 9$

8. Let L be a tangent line to the parabola $y^2 = 4x - 20$ at (6, 2). If L is also a tangent to the ellipse $\frac{x^2}{2} + \frac{y^2}{b} = 1$, then the value of b is equal to :

(1) 20

(2) 14

(3) 16

(4) 11

Ans. **(2)**

Sol. Parabola $y^2 = 4x - 20$

Tangent at P(6, 2) will be

$2y = 4\left(\frac{x+6}{2}\right) - 20$

$2y = 2x + 12 - 20$

$2y = 2x - 8$

$y = x - 4$

$x - y - 4 = 0$ ....(1)

This is also tangent to ellipse $\frac{x^2}{2} + \frac{y^2}{b} = 1$

Apply $c^2 = a^2m^2 + b^2$

$(-4)^2 = (2)(1) + b$

$b = 14$

Option (2)

9. Let $f : R \rightarrow R$ be defined as $f(x) = e^{-x}\sin x$. If $F : [0,1] \ R\rightarrow$ is a differentiable function such that $F(x) \int_0^x f(t)dt$, then the value of $\int_0^1 \left(F'(x) + f(x)\right)e^x dx$ lies in the interval

(1) $\left[\frac{330}{360}, \frac{331}{360}\right]$

(2) $\left[\frac{327}{360}, \frac{329}{360}\right]$

(3) $\left[\frac{331}{360}, \frac{334}{360}\right]$

(4) $\left[\frac{335}{360}, \frac{336}{360}\right]$

Ans. **(1)**

Sol. $F'(x) = f(x)$ by Leibnitz theorem

$\int_0^1 (F'(x) + f(x))e^x\,dx = \int_0^1 2f(x)e^x\,dx$

$I = \int_0^1 2\sin x\,dx$

$I = 2(1 - \cos 1)$

$=\left\{1-\left(1-\frac{1^2}{2!}+\frac{1^4}{4!}-\frac{1}{6!}+\ldots\right)\right\}$

$=2\left\{1-\left(1-\frac{1}{2}+\frac{1}{24}\right)\right\} < 2\ (1 - \cos 1) < 2\left\{1-\left(1-\frac{1}{2}+\frac{1}{24}-\frac{1}{720}\right)\right\}$

$\frac{330}{360} < 2(1 - \cos 1) < \frac{331}{360}$

$\frac{330}{360} < I < \frac{331}{360}$

(1) is correct

10. If x, y, z are in arithmetic progression with common difference d, $x \neq 3d$, and the determinant of the matrix $\begin{bmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{bmatrix}$ is zero, then the value of k^2 is :

(1) 6

(2) 36

(3) 72

(4) 12

Ans. **(3)**

Sol. $\begin{vmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0$

$R_1 \to R_1 + R_3 - 2R_2$

$\begin{vmatrix} 0 & 4\sqrt{2}-k-10\sqrt{2} & 0 \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0 \qquad \{\because\ 2y = x + z\}$

$\Rightarrow \quad (k - 6\sqrt{2})(4z - 5y) = 0$

$k = 6\sqrt{2}$ or $4z = 5y$ (Not possible $\because$ x, y, z in A.P.)

So $\quad k^2 = 72$

$\therefore \quad$ Option (3)

11. If the integral $\int_0^{10} \frac{|\sin 2\pi x|}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$, where α, β, γ are integers and [x] denotes the greatest integer less than or equal to x, then the value of $\alpha + \beta + \gamma$ is equal to :

(1) 20 (2) 0

(3) 25 (4) 10

Ans. **(2)**

Sol. Given integral

$$\int_0^{10}\frac{[\sin 2\pi x]}{e^{x-[x]}}dx = 10\int_0^1\frac{[\sin 2\pi x]}{e^{\{x\}}}dx \quad \text{(using property of definite in.)}$$

$$= 10\left[\int_0^{1/2} 0.dx + \int_{1/2}^{1}\frac{-1}{e^x}dx\right]$$

$$= \therefore -10\left[\frac{e^{-x}}{-1}\right]_{1/2}^{1} = 10\left[e^{-1} - e^{-1/2}\right]$$

$$= 10e^{-1} - 10e^{-1/2}$$

comparing with the given relation,

$\alpha = 10$, $\beta = -10$, $\gamma = 0$

$\alpha + \beta + \gamma = 0$.

therefore, the correct answer is (2).

12. Let y = y(x) be the solution of the differential equation

cosx(3sinx + cosx + 3) dy = (1+y sinx(3sinx + cosx + 3))dx, $0 \le x \le \frac{\pi}{2}$, y(0) = 0. Then, $y\left(\frac{\pi}{3}\right)$ is equal to :

(1) $2\log_e\left(\frac{2\sqrt{3}+10}{11}\right)$

(2) $2\log_e\left(\frac{\sqrt{3}+7}{2}\right)$

(3) $2\log_e\left(\frac{3\sqrt{3}-8}{4}\right)$

(4) $2\log_e\left(\frac{2\sqrt{3}+9}{6}\right)$

Ans. (1)

Sol. cosx (3 sin x + cos x + 3) dy = (1 + y sin x (3 sin x + cos x + 3)) dx ...(1)

(3 sin x + cos x + 3) (cos x dy – y sin x dx) = dx

$$\int d(y.\cos x) = \int\frac{dx}{3\sin x + \cos x + 3}$$

$$y\cos x = \int\frac{1}{3\left(\frac{2+\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+3}$$

$$y\cos x = \int \frac{\sec^2\frac{x}{2}}{6\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2} + 3 + 3\tan^2\frac{x}{2}}$$

$$y\cos x = \int \frac{\sec^2\frac{x}{2}}{2\tan^2\frac{x}{2} + 6\tan\frac{x}{2} + 4} = \int \frac{\frac{1}{2}\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2} + 3\tan\frac{x}{2} + 2}$$

$$y\cos x = \ln\left|\frac{\tan\frac{x}{2} + 1}{\tan\frac{x}{2} + 2}\right| + c$$

Put n =0 & y = 0

$$C = -\ln\left(\frac{1}{2}\right) = \ln(2)$$

$$y\left(\frac{\pi}{3}\right) = 2\ln\left|\frac{1+\sqrt{3}}{1+2\sqrt{3}}\right| + \ln 2$$

$$= 2\ln\left|\frac{5+\sqrt{3}}{11}\right| + \ln 2$$

$$= 2\ln\left|\frac{2\sqrt{3}+10}{11}\right|$$

13. The value of the limit $\lim_{\theta \to 0} \frac{\tan(\pi\cos^2\theta)}{\sin(2\pi\sin^2\theta)}$ is equal to :

(1) $-\frac{1}{2}$

(2) $-\frac{1}{4}$

(3) 0

(4) $\frac{1}{4}$

Ans. **(1)**

Sol. Given,

$$\lim_{\theta \to 0} \frac{\tan(\pi\cos^2\theta)}{\sin(2\pi\sin^2\theta)}$$

$$= \lim_{\theta \to 0} \frac{\tan(\pi - \pi\sin^2\theta)}{\sin(2\pi\sin^2\theta)} \quad (\therefore \cos^2\theta = 1 - \sin^2\theta)$$

$$= \lim_{\theta \to 0} \frac{-\tan(\pi\sin^2\theta)}{\sin(2\pi\sin^2\theta)} \quad (\therefore \tan(\pi - \theta) = -\tan\theta)$$

$$= \lim_{\theta \to 0} \frac{\frac{-\tan(\pi \sin^2\theta)}{\pi \sin^2\theta}}{\frac{\sin(2\pi \sin^2\theta)}{2\pi \sin^2\theta} \times 2} \quad \begin{pmatrix} \text{As } \theta \to 0 \\ \text{the } \sin^2\theta \to 0 \end{pmatrix}$$

$$= \frac{1}{2}. \qquad \left(\because \lim_{\theta \to 0} \frac{\tan\theta}{\theta} \to 1 \;\; \& \;\; \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \right)$$

Therefore, the correct answer is (1).

14. If the curve y = y(x) is the solution of the different equation $2\left(x^2 + x^{5/4}\right)dy - y\left(x + x^{1/4}\right)dx = 2x^{9/4}dx$, x > 0 which passes through the point $\left(1, 1 - \frac{4}{3}\log_e 2\right)$, then the value of y(16) is equal to :

(1) $\left(\frac{31}{3} - \frac{8}{3}\log_e 3\right)$

(2) $4\left(\frac{31}{3} + \frac{8}{3}\log_e 3\right)$

(3) $\left(\frac{31}{3} + \frac{8}{3}\log_e 3\right)$

(4) $4\left(\frac{31}{3} - \frac{8}{3}\log_e 3\right)$

Ans. (4)

Sol. $\frac{dy}{dx} - \frac{y}{2x} = \frac{x^{9/4}}{x^{5/4}(x^{3/4}+1)}$

If = $e^{-\int \frac{ds}{2d}} = e^{-\frac{1}{2}\ln x} = \frac{1}{x^{1/2}}$

$y.x^{-1/2} = \int \frac{x^{9/4}.x^{-1/2}}{x^{5/4}(x^{3/4}+1)}dx$

$\int \frac{x^{1/2}}{(x^{3/4}+1)}dx$

$x = t^4 \Rightarrow dx = 4t^3dt$

$\int \frac{t^2.4t^3dt}{(t^3+1)}$

$4\int \frac{t^2(t^3+1-1)}{(t^3+1)}dt$

$4\int t^2dt - 4\int \frac{t^2}{t^3+1}dt$

$\frac{4t^3}{3} - \frac{4}{3}\ln(t^3+1) + C$

$yx^{-1/2} = \frac{4x^{3/4}}{3} - \frac{4}{3}\ln(x^{3/4} + 1) + C$

$1 - \frac{4}{3}\log_e 2 = \frac{4}{3} - \frac{4}{3}\log_e 2 + C$

$\Rightarrow\ C = -\frac{1}{3}$

$$y = \frac{4}{3}x^{5/4} - \frac{4}{3}\sqrt{x}\ln(x^{3/4} + 1) - \frac{\sqrt{x}}{3}$$

$$y(16) = \frac{4}{3} \times 32 - \frac{4}{3} \times 4\ln 9 - \frac{4}{3}$$

$$= \frac{124}{3} - \frac{32}{3}\ln 3 = 4\left(\frac{31}{3} - \frac{8}{3}\ln 3\right)$$

15. Let S_1, S_2 and S_3 be three sets defined as

$S_1 = \{z \in C : |z - 1| \le \sqrt{2}\}$

$S_2 = \{z \in C : \text{Re}((1 - i)z) \ge 1\}$

$S_3 = \{z \in C : \text{Im}(z) \le 1\}$

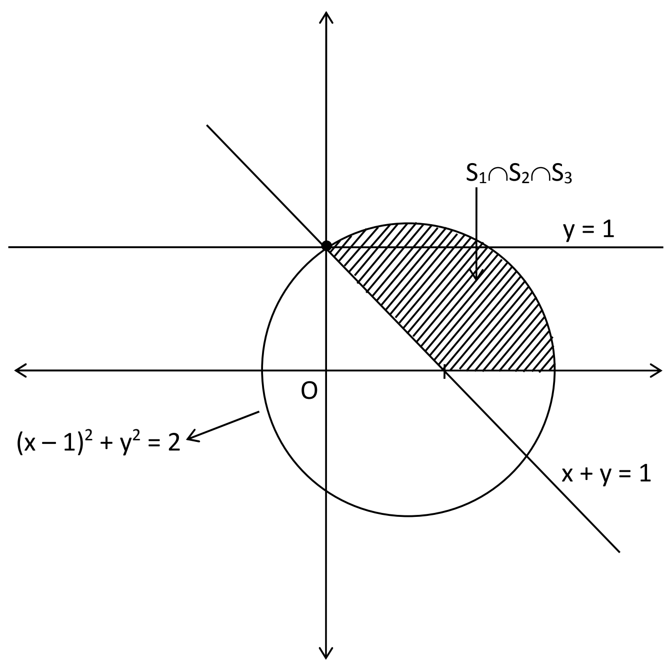
Then the set $S_1 \cap S_2 \cap S_3$

(1) has infinitely many elements (2) has exactly two elements

(3) has exactly three elements (4) is a singleton

Ans. **(1)**

Sol. Let, z = x + iy



$S_1 \equiv (x - 1)^2 + y^2 \le 2$ ...(1)

$S_2 \equiv x + y \ge 1$ ...(2)

$S_3 \equiv y \le 1$ ...(3)

$\Rightarrow S_1 \cap S_2 \cap S_3$ has infinitely many elements.

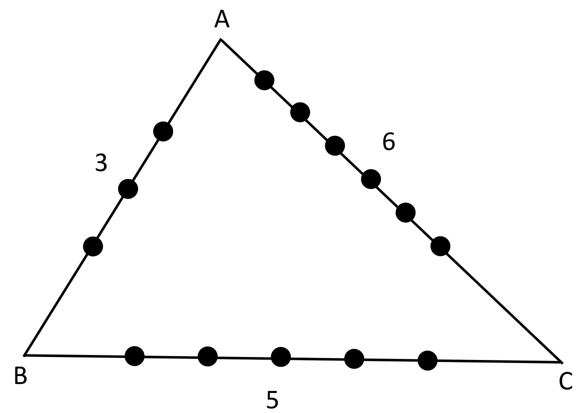
16. If the sides AB, BC, and CA of a triangle ABC have, 3, 5 and 6 interior points respectively, then the total number of triangles that can be constructed using these points as vertices, is equal to:

(1) 360 (2) 240

(3) 333 (4) 364

Ans. (3)

Sol.



Total number of triangles

$= {}^3C_1 \times {}^5C_1 \times {}^6C_1$

$+ {}^3C_1 \times {}^5C_2 + {}^5C_1 \times {}^3C_2$

$+ {}^3C_1 \times {}^6C_2 + {}^6C_1 \times {}^3C_2$

$+ {}^5C_1 \times {}^6C_2 + {}^6C_1 \times {}^5C_2$

$= 90 + 30 + 15 + 45 + 18 + 75 + 60$

$= 333$

17. The value of

$$\lim_{x \to \infty} \frac{[r] + [2r] + \ldots + [nr]}{n^2},$$

Where r is a non-zero real number and [r] denotes the greatest integer less than or equal to r, is equal to :

(1) 0

(2) r

(3) $\frac{r}{2}$

(4) 2r

Ans. (3)

Sol. We know,

$(x - 1) \le [x] < x$

$\therefore (r - 1) \le [r] < r \to r$

$(2r - 1) \leq [2r] < 2r \rightarrow r$

.

.

.

$(nr - 1) \leq [nr] < nr$

Adding

$$\frac{n(n+1)}{2}r - n \leq [r] + [2r] + \ \ldots\ldots [nr] < \frac{n(n+1)}{2}r$$

$$\lim_{n\to\infty}\left(\frac{\frac{n(n+1)}{2}r - n}{n^2}\right) \leq L < \lim_{n\to\infty}\frac{n(n+1)}{2}r$$

$$\Rightarrow \frac{r}{2} \leq L < \frac{r}{2}$$

$$\Rightarrow L = \frac{r}{2}$$

18. The value of $\sum_{r-0}^{6}\left({}^{6}C_{r} \cdot {}^{6}C_{6-r}\right)$ is equal to :

(1) 1124 (2) 924

(3) 1324 (4) 1024

Ans. (2)

Sol. Given,

$$\sum_{r=0}^{6} {}^{6}C_{r} {}^{6}C_{6-r}$$

$$= \sum_{r=0}^{6} {}^{6+6}C_{r+6-r}$$

$$= \sum_{r=0}^{6} {}^{12}C_{6}$$

$$= \frac{12!}{6!6!} = 924$$

Therefore, the correct answer is (2).

19. Two tangents are drawn from a point P to the circle $x^2 + y^2 - 2x - 4y + 4 = 0$, such that the angle between these tangents is $\tan^{-1}\left(\frac{12}{15}\right)$, where $\tan^{-1}\left(\frac{12}{5}\right) \in (0, \pi)$. If the centre of the circle is denoted by C and these tangents touch the circle at points A and B, then the ratio of the areas of ΔPAB and ΔCAB is :

(1) 11 : 4

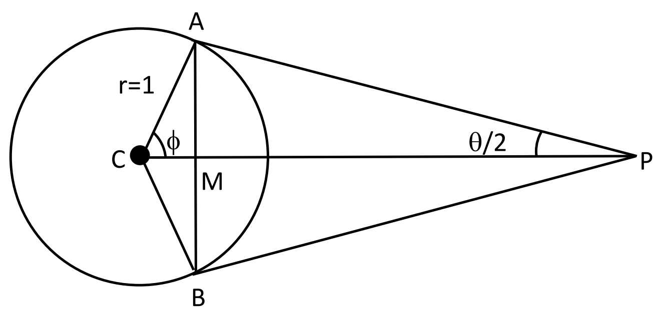
(2) 9 : 4

(3) 2 : 1

(4) 3 : 1

Ans. (2)

Sol.



Let $\theta = \tan^{-1}\left(\frac{12}{5}\right)$

$\Rightarrow \tan\theta = \frac{12}{5}$

$\Rightarrow \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{12}{5}$

$\Rightarrow \tan\frac{\theta}{2} = \frac{2}{3} \quad \Rightarrow \sin\frac{\theta}{2} = \frac{2}{\sqrt{3}}$ and $\cos\frac{\theta}{2} = \frac{3}{\sqrt{13}}$

In ΔCAP,

$\tan\frac{\theta}{2} = \frac{1}{AP}$

$\Rightarrow AP = \frac{3}{2}$

In ΔAPM, $\sin\frac{\theta}{2} = \frac{AM}{AP}$, $\cos\frac{\theta}{2} = \frac{PM}{AP}$

$\Rightarrow AM = \frac{3}{\sqrt{13}} \quad \Rightarrow PM = \frac{9}{2\sqrt{13}}$

$\therefore AB = \frac{6}{\sqrt{13}}$

$\therefore$ Area of $\Delta PAB = \frac{1}{2} \times AB \times PM$

$= \frac{1}{2} \times \frac{6}{\sqrt{13}} \times \frac{9}{2\sqrt{13}} = \frac{27}{26}$

Now, $\phi = 90^o - \frac{\theta}{2}$.

In ΔCAM,

$\cos\phi = \dfrac{CM}{CA}$

$\Rightarrow$ $CM = 1.\cos\left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right)$

$= 1.\sin\dfrac{\theta}{2} = \dfrac{2}{\sqrt{13}}$

$\therefore$ Area of $\Delta CAB = \dfrac{1}{2} \times AB \times CM$

$= \dfrac{1}{2} \times \dfrac{6}{\sqrt{13}} \times \dfrac{2}{\sqrt{13}} = \dfrac{6}{13}$

$\therefore$ $\dfrac{\text{Area of } \Delta PAB}{\text{Area of } \Delta CAB} = \dfrac{27/26}{6/13} = \dfrac{9}{4}$

Therefore, the correct answer is (2).

20. The number of solutions of the equation $\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$, for $x \in [-1, 1]$, and [x] denotes the greatest integer less than or equal to x, is :

(1) 0

(2) 2

(3) 4

(4) Infinite

Ans. **(1)**

Sol. There are three cases possible for $x \in [-1, 1]$

Case I : $x \in \left[-1, -\sqrt{\frac{2}{3}}\right)$

$\therefore$ $\sin^{-1}(1) + \cos^{-1}(0) = x^2$

$\Rightarrow$ $x^2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ $\Rightarrow$ $x = \pm\sqrt{\pi}$ $\rightarrow$ (Reject)

Case II : $x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$

$\therefore$ $\sin^{-1}(0) + \cos^{-1}(-1) = x^2$

$\Rightarrow$ $0 + \pi = x^2$ $\Rightarrow$ $x = \pm\sqrt{\pi}$ $\rightarrow$ (Reject)

Case III : $x \in \left(\sqrt{\frac{2}{3}}, 1\right)$

$\therefore$ $\sin^{-1}(0) + \cos^{-1}(0) = x^2$

$\Rightarrow$ $x^2 = \pi$ $\Rightarrow$ $x = \pm\sqrt{\pi}$ (Reject)

$\therefore$ No solution. There, the correct answer is (1).

SECTION - B

1. Let the coefficients of third, fourth and fifth terms in the expansion of $\left(x+\frac{a}{x^2}\right)^n$, $x \neq 0$, be in the ration 12:8:3. Then the term independent of *x* in the expansion, is equal to ....

Ans. **(4)**

Sol. $T_{r+1} = n_{cr}x^{n-r}.\left(\frac{a}{x^2}\right)^r$

$= {}^nc_r a^r x^{n-3r}$

$T_3 = {}^nc_2 a^2 x^{n-6}$, $T_4 = {}^nc_3 a^3 x^{n-9}$

$T_5 = {}^nc_4 a^4 x^{n-12}$

Now, $\frac{\text{coefficient of } T_3}{\text{coefficient of } T_4} = \frac{{}^nc_4.a^2}{{}^nc_3.a^3} = \frac{3}{a(n-2)} = \frac{3}{2}$

$\Rightarrow$ a(n– 2) = 2 .... (i)

and $\frac{\text{coefficient of } T_4}{\text{coefficient of } T_5} = \frac{{}^nc_3.a^3}{{}^nc_4.a^4} = \frac{4}{a(n-3)} = \frac{8}{3}$

$\Rightarrow$ a(n– 3) = $\frac{3}{2}$ .... (ii)

by (i) and (ii) n = 6, a = $\frac{1}{2}$

for term independent of 'x'

$n - 3r = 0 \Rightarrow r = \frac{n}{3} \Rightarrow r = \frac{6}{3} = 2$

$T_3 = {}^6c_2\left(\frac{1}{2}\right)^2 x^0 = \frac{15}{4} = 3.75 \approx 4$

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that AB = B and a + d = 2021, then the value of ad-bc is equal to ....

Ans. **(2020)**

Sol. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

AB = B

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$\begin{bmatrix} a\alpha + b\beta \\ c\alpha + d\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow$ $a\alpha + b\beta = \alpha$ ....(1)
and $c\alpha + d\beta = \beta$ ....(2)

$\alpha(a-1) = -b\beta$ and $c\alpha = \beta(1-d)$

$\frac{\alpha}{\beta} = \frac{-b}{a-1}$ & $\frac{\alpha}{\beta} = \frac{1-d}{c}$

$\therefore \frac{-b}{a-1} = \frac{1-d}{c}$

$-bc = (a-1)(1-d)$

$-bc = a - ad - 1 + d$

$ad - bc = a + d - 1$

$= 2021 - 1$

$= 2020$

3. Let $f : [-1,1] \rightarrow R$ be defined as $f(x) = ax^2+bx+c$ for all $x \in [-1, 1]$, where $a, b, c \in R$ such that $f(-1) = 2$, $f'(-1) = 1$ and for $x \in [-1, 1]$ the maximum value of $f''(x)$ is $\frac{1}{2}$. If $f(x) \leq \alpha$, $x \in [-1, 1]$, then the least value of α is equal to ....

Ans. **(5)**

Sol. $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$,

$f''(x) = 2a$

Given $f''(-1) = \frac{1}{2}$ $\Rightarrow a = \frac{1}{4}$

$f'(-1) = 1 \Rightarrow b - 2a = 1$ $\Rightarrow b = \frac{3}{2}$

$f(-1) = a - b + c = 2$ $\Rightarrow c = \frac{13}{4}$

Now $f(x) = \frac{1}{4}(x^2 + 6x + 13)$, $x \in [-1, 1]$

$f'(x) = \frac{1}{4}(2x + 6) = 0$ $\Rightarrow x = -3 \notin [-1, 1]$

$f(1) = 5$, $f(-1) = 2$

$f(x) \leq 5$

So $\alpha_{minimum} = 5$

4. Let $I_n = \int_1^e x^{19}(\log|x|)^n\, dx$, where $n \in N$. If $(20)I_{10} = \alpha I_9 + \beta I_8$, for natural numbers α and β, then $\alpha - \beta$ equal to ....

Ans. **(1)**

Sol. $I_n = 2\int_1^e x^{19}(\ell nx)^n .dx$

$$= (\ell nx)^n . \frac{x^{20}}{20}\Big|_1^e \int_1^e n\frac{(\ell nx)^{n-1}}{x}\frac{x^{20}}{20}dx$$

$$I_n = \frac{e^{20}}{20} - \frac{n}{20}(I_{n-1})$$

$$20I_n = e^{20} - n\, I_{n-1}$$

$$20I_{10} = \left(e^{20} - 10I_9\right) \qquad ....(1)$$

$$20I_9 = e^{20} - 9I_8 \qquad ....(2)$$

– – –

$$20I_{10} = 10I_9 + 9I_8$$

$\alpha = 10$, $\beta = 9$ $\Rightarrow$ $\alpha - \beta = 1$

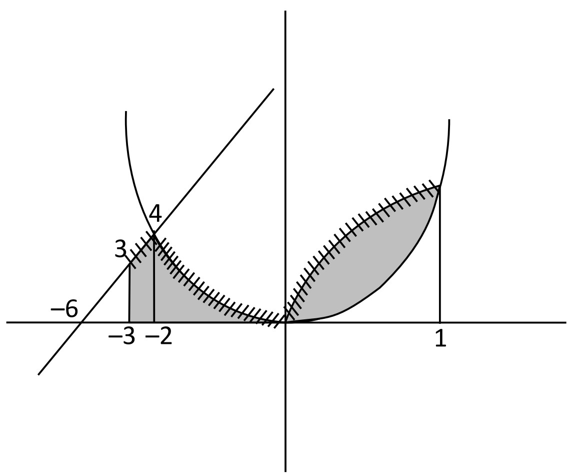
5. Let $f : [-3, 1] \rightarrow R$ be given as

$$f(x) = \begin{cases} \min\{(x+6), x^2\}, & -3, \le x \le 0 \\ \max\{\sqrt{x}, x^2\}, & 0 \le x \le 1. \end{cases}$$

If the area bounded by y = *f*(x) and x-axis is A, then the value of 6A is equal to …………… .

Ans. **(41)**

Sol.



Area is $\int_{-3}^{-2} (x+6)dx + \int_{-2}^{0} x^2 dx + \int_{0}^{1} \sqrt{x} dx = A$

$$= \frac{7}{2} + \left[\frac{x^3}{3}\right]_{-2}^{0} + \left[\frac{2}{3}x^{3/2}\right]_{0}^{1}$$

$$= \frac{7}{2} + \frac{8}{3} + \frac{2}{3} = \frac{41}{6}$$

So, 6A = 41

6. Let $\vec{x}$ be a vector in the plane containing vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$. If the vector $\vec{x}$ is perpendicular to $(3\hat{i} + 2\hat{j} - \hat{k})$ and its projection on $\vec{a}$ is $\frac{17\sqrt{6}}{2}$, then the value of $|\vec{x}|^2$ is equal to ……………….. .

Ans. (486)

Sol. Let, $\vec{x} = k(\vec{a} + \lambda\vec{b})$

$x \to$ is perpendicular to $3\hat{i} + 2\hat{j} - \hat{k}$

I. $k\{(2 + \lambda)3 + (2\lambda - 1)2 + (1 - \lambda)(-1) = 0$

$\Rightarrow 8\lambda + 3 = 0$

$$\lambda = \frac{-3}{8}$$

II. Also projection of $\vec{x}$ on $\vec{a}$ is therefore

$$\frac{\vec{\alpha}.\vec{a}}{|\vec{a}|} = \frac{17\sqrt{6}}{2}$$

$$\Rightarrow k\left\{\frac{(\vec{a} + \lambda\vec{b}).\vec{a}}{\sqrt{6}}\right\} = \frac{17\sqrt{6}}{2}$$

$$\Rightarrow k\left\{6 + \left(\frac{3}{8}\right)\right\} = \frac{17 \times 6}{2}$$

$$\Rightarrow k = \frac{51}{51} \times 8$$

$k = 8$

$$\vec{x} = 8\left(\frac{13}{8}\hat{i} - \frac{14}{8}\hat{j} + \frac{11}{8}\hat{k}\right)$$

$$= 13\hat{i} - 14\hat{j} + 11\hat{k}$$

$|\vec{x}|^2 = 169 + 196 + 121 = 486$

7. Consider a set of 3n numbers having variance 4. In this set, the mean of first 2n numbers is 6 and the mean of the remaining n numbers is 3. A new set is constructed by adding 1 into each of first 2n numbers, and subtracting 1 from each of the remaining n numbers. If the variance of the new set is k, then 9k is equal to ………………. .

Ans. (68)

Sol. Let first 2n observations ae x_1, x_2 ……….. , x_{2n0}

and last n observations are y_1, y_2 ……….. , y_n

Now, $\frac{\sum x_i}{2n} = 6$, $\frac{\sum y_i}{n} = 3$

$\Rightarrow \sum x_i = 12n$, $\sum y_i = 3n$ $\quad \therefore \frac{\sum x_i + \sum y_i}{3n} = \frac{15n}{3n} = 5$

Now, $\frac{\sum x_i^2 + \sum y_i^2}{3n} - 5^2 = 4$

$\Rightarrow \sum x_i^2 + \sum y_i^2 = 29 \times 3n = 87n$

Now, mean is $\frac{\sum(x_i+1)+\sum(y_i-1)}{3n}=\frac{15n+2n-n}{3n}=\frac{16}{3}$

Now, variance is $\frac{\sum(x_i+1)^2+\sum(y_i-1)^2}{3n}-\left(\frac{16}{3}\right)^2$

$=\frac{\sum x_i^2+\sum y_i^2+2\left(\sum x_i-\sum y_i\right)+3n}{3n}-\left(\frac{16}{3}\right)^2$

$=\frac{87n+2(9n)+3n}{3n}-\left(\frac{16}{3}\right)^2$

$29+6+1-\left(\frac{16}{3}\right)^2$

$=\frac{324-256}{9}=\frac{68}{9}=k$

$\Rightarrow \boxed{9k=68}$

Therefore, the correct answer is 68.

8. If 1, $\log_{10}(4^x-2)$ and $\log_{10}\left(4^x+\frac{18}{5}\right)$ are in arithmetic progression for a real number x, then the value of the determinant $\begin{vmatrix} 2\left(x-\frac{1}{2}\right) & x-1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$ is equal to :

Ans. **(2)**

Sol. 1, $\text{Log}_{10}\left(4^x-2\right)$, $\text{Log}_{10}\left(4^x+\frac{18}{5}\right)$ in AP.

$2.\ \text{Log}_{10}\left(4^x-2\right)=1+\text{Log}_{10}\left(4^x+\frac{18}{5}\right)$

$\text{Log}_{10}\left(4^x-2\right)^2=\log_{10}\left(10.\left(4^x+\frac{18}{5}\right)\right)$

$\left(4^x-2\right)^2=10.\left(4^x+\frac{18}{5}\right)$

$\left(4^x\right)^2+4-4.4^x=10.4^x+36$

$\left(4^x\right)^2-14.4^x-32=0$

$\left(4^x\right)^2+2.4^x-16.4^x-32=0$

$4^x\left(4^x+2\right)-16.\left(4^x+2\right)=0$

$\left(4^x+2\right)\left(4^x-16\right)=0$

$4^x=-2 \qquad 4^x=16$

$\times \qquad x = 2$

Therefore $\begin{vmatrix} 2(x-1/2) & x-1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 3 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix}$

$= 3(-2) - 1(0-4) + 4(1-0)$

$= -6 + 4 + 4$

$= 2$

9. Let P be an arbitrary point having sum of the squares of the distances from the planes x + y + z = 0, *l*x − nz = 0 and x − 2y + z = 0, equal to 9. If the locus of the point P is $x^2 + y^2 + z^2 = 9$, then the value of *l*–n is equal to …………….. .

Ans. **(0)**

Sol. Let point P is (α, β, γ)

$$\left(\frac{\alpha+\beta+\gamma}{\sqrt{3}}\right)^2 + \left(\frac{\ell\alpha - n\gamma}{\sqrt{\ell^2+n^2}}\right)^2 + \left(\frac{\alpha-2\beta+\gamma}{\sqrt{6}}\right)^2 = 9$$

Locus is $\frac{(x+y+z)^2}{3} + \frac{(\ell n - nz)^2}{\ell^2+n^2} + \frac{(x-2y+z)^2}{6} = 9$

$$x^2\left(\frac{1}{2}+\frac{\ell^2}{\ell^2+n^2}\right) + y^2 + z^2\left(\frac{1}{2}+\frac{n^2}{\ell^2+n^2}\right) + 2zx\left(\frac{1}{2}-\frac{\ell n}{\ell^2+n^2}\right) - 9 = 0$$

Since its given that $x^2 + y^2 + z^2 = 9$

After solving $\ell = n$,

then $\ell - n = 0$

10. Let $\tan\alpha$, $\tan\beta$ and $\tan\gamma$; $\alpha, \beta, \gamma \neq \frac{(2n-1)\pi}{2}$, $n \in N$ be the slopes of three line segment OA, OB and OC, respectively, where O is origin. If circumcentre of ΔABC coincides with origin and its orthocentre lies on y-aixs, then the value of $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}\right)^2$ is equal to

Ans. **(144)**

Sol. Since orthocentre and circumcentre both lies on y-axis

$\Rightarrow$ Centroid also lies on y-axis

$\Rightarrow \Sigma\cos\alpha = 0$

$\cos\alpha + \cos\beta + \cos\gamma = 0$

$\Rightarrow \cos^3\alpha + \cos^3\beta + \cos^3\gamma = 3\cos\alpha\cos\beta\cos\gamma$

$$\therefore \frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha \cos\beta \cos\gamma}$$

$$= \frac{4\left(\cos^3\alpha + \cos^3\beta + \cos^3\gamma\right) - 3\left(\cos\alpha + \cos\beta + \cos\gamma\right)}{\cos\alpha \cos\beta \cos\gamma} = 12$$

then, $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha \cos\beta \cos\gamma}\right)^2 = 144$