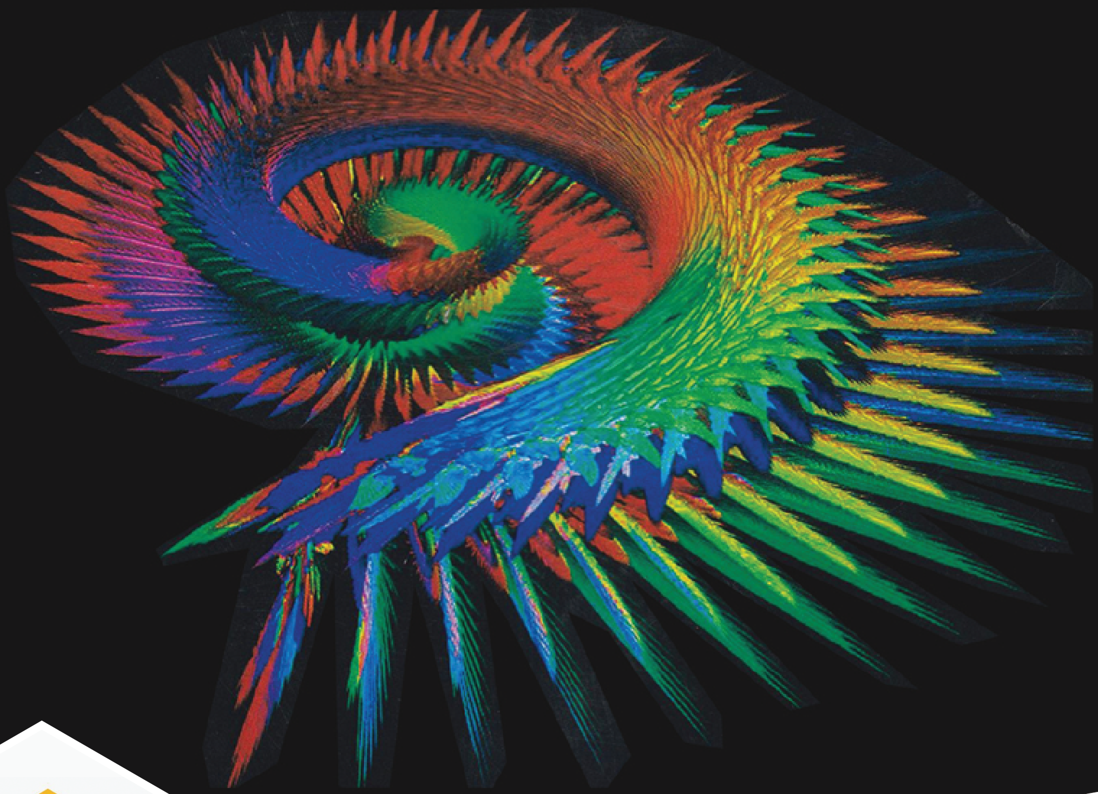
Telangana State Board of
INTERMEDIATE Education

MATHEMATICS-IIB



BASIC LEARNING MATERIAL

For The Academic Year : 2021-2022

TELANGANA STATE BOARD OF INTERMEDIATE EDUCATION

MATHEMATICS-II B

(ENGLISH MEDIUM)

BASIC LEARNING MATERIAL

ACADEMIC YEAR

2021-2022

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PREFACE

The ongoing Global Pandemic Covid-19 that has engulfed the entire world has changed every sphere of our life. Education, of course is not an exception. In the absence and disruption of Physical Classroom Teaching, Department of Intermediate Education Telangana has successfully engaged the students and imparted education through TV lessons. In the back drop of the unprecedented situation due to the pandemic TSBIE has reduced the burden of curriculum load by considering only 70% syllabus for class room instruction as well as for the forthcoming Intermediate Examinations. It has also increased the choice of questions in the examination pattern for the convenience of the students.

To cope up with exam fear and stress and to prepare the students for annual exams in such a short span of time , TSBIE has prepared “Basic Learning Material” that serves as a primer for the students to face the examinations confidently. It must be noted here that, the Learning Material is not comprehensive and can never substitute the Textbook. At most it gives guidance as to how the students should include the essential steps in their answers and build upon them. I wish you to utilize the Basic Learning Material after you have thoroughly gone through the Text Book so that it may enable you to reinforce the concepts that you have learnt from the Textbook and Teachers. I appreciate ERTW Team, Subject Experts, who have involved day in and out to come out with the Basic Learning Material in such a short span of time.

I would appreciate the feedback from all the stake holders for enriching the learning material and making it cent percent error free in all aspects.

The material can also be accessed through our website**www.tsbie.cgg.gov.in.**

Commissioner & Secretary
Intermediate Education, Telangana.

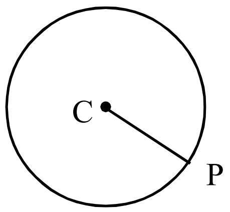
CONTENTS

Unit - 1	**Circle**	**01 - 62**
Unit - 2	**System of Circles**	**63 - 84**
Unit - 3	**Parabola**	**85 - 102**
Unit - 4	**Ellipse**	**103 - 116**
Unit - 5	**Hyperbola**	**117 - 126**
Unit - 6	**Integration**	**127 - 154**
Unit - 7	**Definite Integrals**	**155 - 175**
Unit - 8	**Differential Equations**	**176 - 192**

Unit **1**

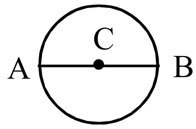
Circle

Definition : A circle is a set of points in a plane such that they are equidistant from a fixed point lying in the plane.



C is the centre, CP = radius

The fixed point is called the centre and the distance from the centre to any point on the circle is called the radius of the circle.

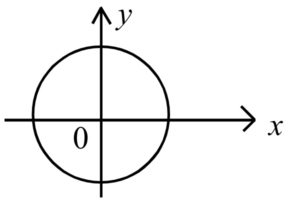


AB = 2CB = (2 x radius) is called the diameter of the circle

The equation of the circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$

If the centre (a, b) is origin, i.e., $(a, b) = (0, 0)$, then the equation of the circle with radius r is

$\boxed{x^2 + y^2 = r^2}$



- The standard equation or General equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ whose centre is $(-g, -f)$, radius $= r = \sqrt{g^2 + f^2 - c}$
- The equation of the circle whose extremities of diameter are (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

- The parametric equations of the circle are

$$x = x_1 + r\cos\theta$$

$$y = y_1 + r\sin\theta$$

Where (x_1, y_1) = centre and r = radius of the circle,

θ is the parameter and $0 \le \theta < 2\pi$

Note : The parametric equations of a circle describe the coordinates of a point (x, y) on the circle in terms of a single variable 'θ' and 'θ' is called as parameter.

- So any point on the circle is given by

$$(x, y) = (x_1 + r\cos\theta, y_1 + r\sin\theta) = \text{'point } \theta\text{'}$$

called as 'point θ' where (x_1, y_1) is the centre and 'r' is the radius of the circle

- The parametric equations of the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ are

$$x = -g + r\cos\theta$$

$$y = -f + r\sin\theta$$

Where $r = \sqrt{g^2 + f^2 - c}$

Any 'point θ' on the circle is 'point θ' = (x, y)

$$= (-g + r\cos\theta, -f + r\sin\theta)$$

- The parametric equations of a circle with centre origin and radius 'r' is

$$x = r\cos\theta$$

$$y = r\sin\theta, \qquad 0 \le \theta < 2\pi$$

- The second order non - homogeneous equation in x and y, that is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle iff

(i) $a = b \ne 0$ (coeff of x^2 = coeff of y^2)

(ii) $h = 0$ (coeff of xy is zero)

(iii) $g^2 + f^2 - ac \ge 0$

Notation

$$S = x^2 + y^2 + 2gx + 2fy + c$$

$$S_1 = x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c$$

$$S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$S_{12} = x_1 x_2 + y_1 y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c$

$S_{21} = S_{12}$

$S_{22} = x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c$

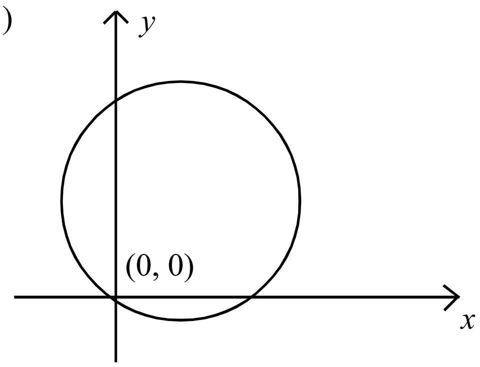
Note : $S_{11} = S/_{(x_1, y_1)} = S_1/_{(x_1, y_1)}$

$S_{12} = S_1/_{(x_2, y_2)} = S_2/_{(x_1, y_1)}$

So, S = 0 represents a circle

S = 0 means $x^2 + y^2 + 2gx + 2fy + c = 0$

- Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ ...... (1) pass through origin (0, 0)



$\Rightarrow$ (0, 0) should satisfy (1)

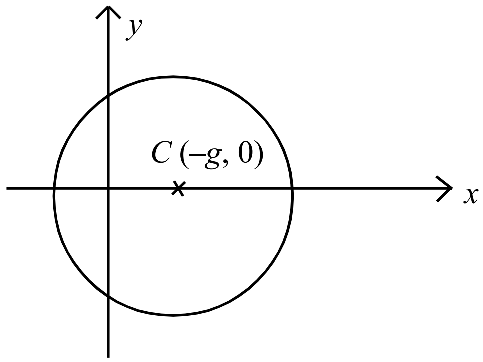
$\because$ (0, 0) is a point on the circle

$\Rightarrow$ $0^2 + 0^2 + 2g(0) + 2f(0) + c = 0$

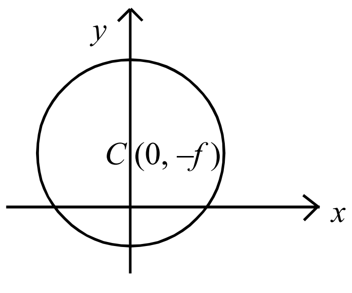
$\Rightarrow$ $\boxed{c = 0}$

$\therefore$ The circle passing through origin is of the form $x^2 + y^2 + 2gx + 2fy = 0$

- If the centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ lies on x - axis then $(-g, -f)$ lies on x - axis $\Rightarrow$ $\boxed{f = 0}$ because every point on x - axis have its y - coordinate as zero

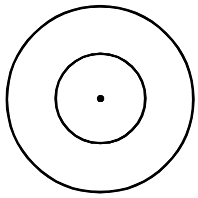


- If the centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ lies on y - axis then $(-g, -f)$ lies on y - axis. $\Rightarrow$ $\boxed{g = 0}$ because every point on y - axis have its x coordinate as zero.

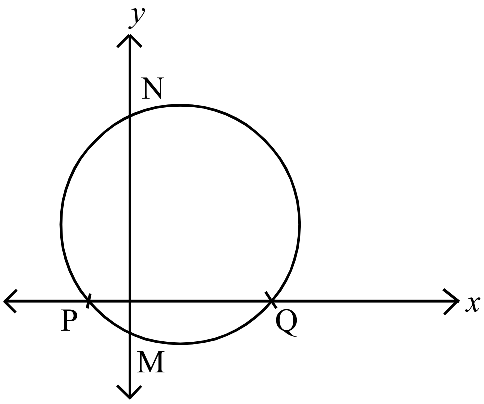


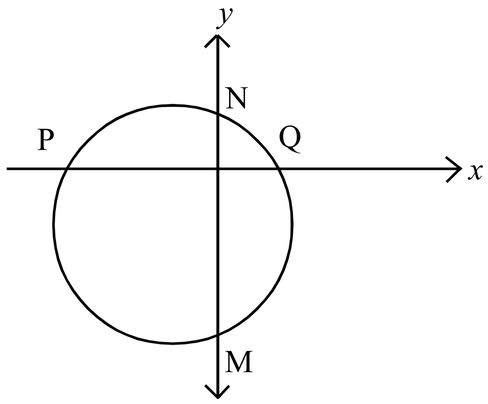
- Two or more circles are said to be concentric if their centres are same.

  Note : The equation of any circle concentric with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is of the form $x^2 + y^2 + 2gx + 2fy + c^1 = 0$ where c^1 is a constant. Their centres are same.



- If the radius of the circle is one, then it is called as unit circle.
- If the circle intersects x - axis at 'P' and 'Q' then the distance PQ is called as x - intercept made by the circle on x - axis.
- If the circle intersects y - axis at 'M' and 'N' then the distance MN is called as y - intercept made by the circle on y - axis





PQ is x - intercept MN is y - intercept

- If $(g^2 - c) > 0$, then the intercept made on the x-axis by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $2\sqrt{g^2 - c}$.

  $\therefore$ PQ $= 2\sqrt{g^2 - c}$

  x - intercept = length of chord PQ = DistancePQ = $2\sqrt{g^2 - c}$

- If the x - axis touches the circle, then P and Q coincide i.e., length of chord PQ is zero or x-intercept is zero

$\Rightarrow$ $2\sqrt{g^2 - c} = 0 \Rightarrow g^2 - c = 0$

$\therefore$ The condition for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to touch the x - axis is $g^2 - c = 0$ or $g^2 = c$

- If $(f^2 - c) > 0$, then the intercept made on the y - axis by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $2\sqrt{f^2 - c}$

$\text{MN} = 2\sqrt{f^2 - c}$

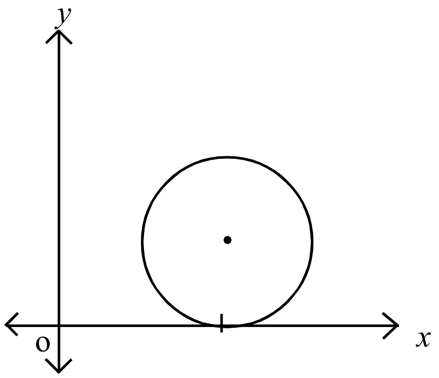
y - intercept $= 2\sqrt{f^2 - c}$

- If the y - axis touches the circle then M and N coincide, the length of chord MN is zero

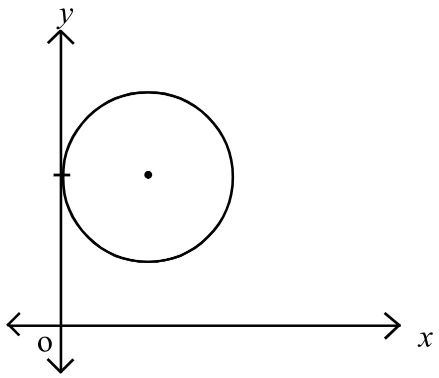
  or y - intercept is zero $\Rightarrow \quad 2\sqrt{f^2 - c} \;\Rightarrow f^2 - c = 0$

  $\therefore$ The condition for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ to touch the

  y - axis is $f^2 - c = 0$ or $f^2 = c$.



circle touches x - axis $\Rightarrow g^2 = c$



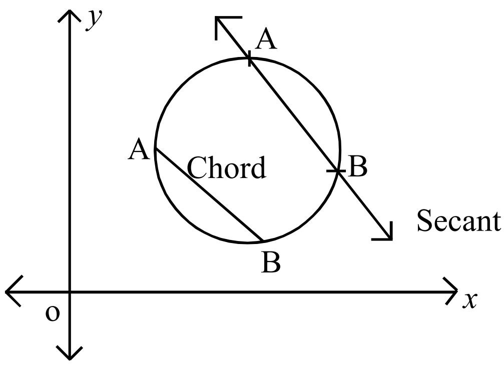
circle touches y - axis $\Rightarrow f^2 = c$

Definition :

If A and B are two distinct points on a circle, then

(i) The line $\overleftrightarrow{\text{AB}}$ through A and B is called a secant.

(ii) The segment $\overline{\text{AB}}$ is called a chord. The length of the chord is denoted by $\overline{\text{AB}}$.



Notation :

Let P (x_1, y_1)

If $\quad S = x^2 + y^2 + 2gx + 2fy + c = 0$

then $\quad S_1 = x\,x_1 + y\,y_1 + g(x + x_1) + f\,(y + y_1) + c$

Example : If $S = x^2 + y^2 + 3x - 5y + 9$, $(x_1, y_1) = (-2, 3)$

Then $S_1 = x(-2) + y(3) + \frac{3}{2}(x-2) - \frac{5}{2}(y+3) + 9$ $[\because 2g = 3, 2f = -5]$

$$= -2x + 3y + \frac{3x-6}{2} - \frac{(5y+15)}{2} + 9$$

$$= \frac{-4x + 6y + 3x - 6 - 5y - 15 + 18}{2}$$

$$= \frac{-x + y - 3}{2}$$

$S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\therefore$ S_{11} for the above circle is 'S' value at (x_1, y_1)

$\therefore$ $S_{11} = (-2)^2 + 3^2 + 3(-2) - 5(3) + 9$

$= 4 + 9 - 6 - 15 + 9 = 1$

So, S_1 is a first degree expression in x & y.

S_{11} is a real number.

- **Important Note :** While writing S_1 or S_{11}, first write S = 0 in the standard form i.e., if the circle is $3x^2 + 3y^2 + 4x + 5y + 7 = 0$ then $S = x^2 + y^2 + \frac{4}{3}x + \frac{5}{3}y + \frac{7}{3}$.

  $S_1 = xx_1 + yy_1 + \frac{2}{3}(x + x_1) + \frac{5}{6}(y + y_1) + \frac{7}{3}$

- **Position of a point with respect to a circle**

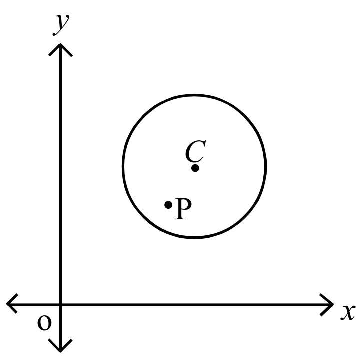
  Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ be a circle in a plane and $P(x_1, y_1)$ be any point in the same plane.

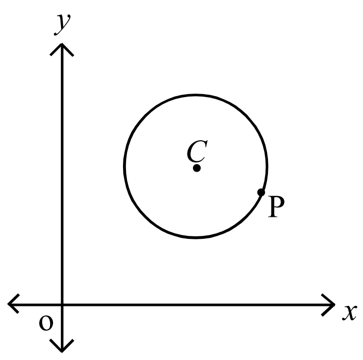
Then

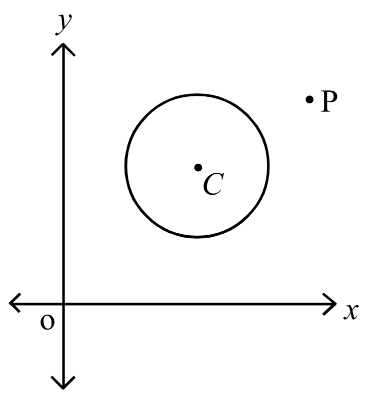
(i) $P(x_1, y_1)$ lies in the interior of the circle, $\Leftrightarrow$ $S_{11} < 0$

(ii) $P(x_1, y_1)$ lies on the circle, $\Leftrightarrow$ $S_{11} = 0$

(iii) $P(x_1, y_1)$ lies in the exterior of the circle, $\Leftrightarrow$ $S_{11} > 0$

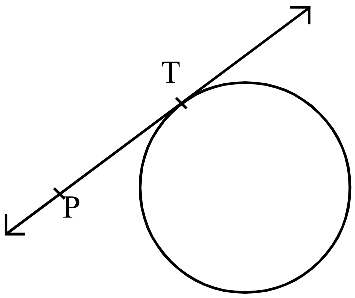






P lies inside the circle $\Leftrightarrow$ $S_{11} < 0$ P lies on the circle $\Leftrightarrow$ $S_{11} = 0$ P lies outside the circle $\Leftrightarrow$ $S_{11} > 0$

- **Length of the tangent from P(x_1, y_1) to the circle**



Tangent with respect to a circle is a straight line, which touches the circle at one point.

In the above figure the line $\overleftrightarrow{PT}$ is a tangent to the circle at T and T is called as the point of contact of tangent to the circle.

If P is an external point to the circle S = 0 where S $= x^2 + y^2 + 2gx + 2fy + c$, and PT is a tangent from P(x_1, y_1) to the circle S = 0, then the distance PT is called as the length of the tangent from P to the circle S = 0

It is given by the formula $\sqrt{S_{11}}$.

$\therefore$ PT = Length of tangent from P to the circle S = 0 is $\sqrt{S_{11}}$.

Definition

The power of a point P with respect to the circle, whose centre is 'C' and radius 'r' is defined as the value = (CP$^2 - r^2$)

- The power of the point P(x_1, y_1) with respect to the circle
  S = $x^2 + y^2 + 2gx + 2fy + c = 0$ is S_{11}.
- **Chord, tangent, Normal** $\longrightarrow$ equations in different form.

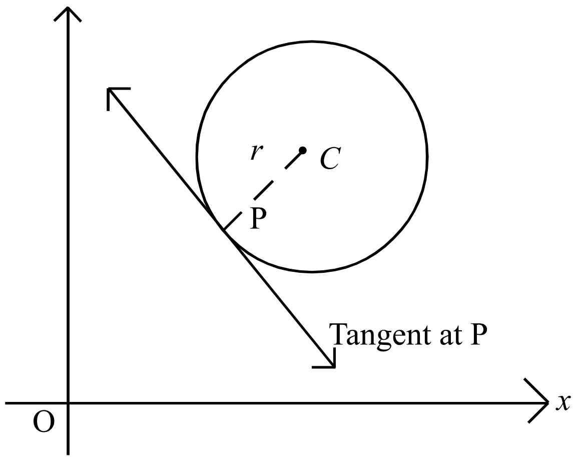
Chord

- If A(x_1, y_1), B(x_2, y_2) are two points on the circle S = 0, then the equation of the secant $\overleftrightarrow{AB}$ or chord $\overline{AB}$ is $S_1 + S_2 = S_{12}$
- If 'point θ_1' = ($-g + r\cos\theta_1$, $-f + r\sin\theta_1$) and 'point θ_2' = ($-g + r\cos\theta_2$, $-f + r\sin\theta_2$) are two points on the circle S = 0 where $r = \sqrt{g^2 + f^2 - c}$, then the equation of the chord joining these two points is

$$(x+g)\cos\left(\frac{\theta_1+\theta_2}{2}\right)+(y+f)\sin\left(\frac{\theta_1+\theta_2}{2}\right)=r\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

- The line meets the circle in one and only one point 'P' ie, touches the circle.

  This line is called as Tangent to the circle at the point 'P' on the circle



- The equation of tangent, w.r.t the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is

  (i) $S_1 = 0$ or $x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c = 0$ [point form]

  where (x_1, y_1) is a point on the circle $S = 0$

  (ii) $y + f = m(x+g) \pm r\sqrt{1+m^2}$, in the slope form

  where $r = \sqrt{g^2 + f^2 - c}$ = radius and m is the slope of tangent

  (iii) $(x + g)\cos\theta + (y + f)\sin\theta = r$, in the parameteric form

  where r = radius = $\sqrt{g^2 + f^2 - c}$ and 'point θ' on the circle is

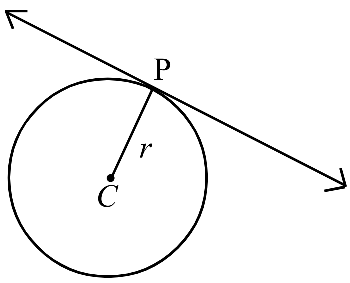
  $(-g + r\cos\theta, -f + r\sin\theta) = (x_1, y_1)$, θ is the parameter

- The equation of tangent w.r.t the circle $S = x^2 + y^2 - r^2 = 0$ is

  (i) $S_1 = 0$ or $x x_1 + y y_1 - r^2 = 0$ where $P(x_1, y_1)$ is a point on the circle $S = (x^2 + y^2 - r^2) = 0$

  (ii) $y = mx \pm r\sqrt{1+m^2}$, in the slope form, where m is the slope of tangent.

  (iii) $x\cos\theta + y\sin\theta = r$; in the parameteric form at point 'θ' = $(r\cos\theta, r\sin\theta)$, on the circle.



Condition for tangency

The condition for a line

$L = lx + my + n = 0$ to touch the circle

$S = x^2 + y^2 + 2gx + 2fy + c = 0$ is

radius = perpendicular distance from the centre C to the line L = 0

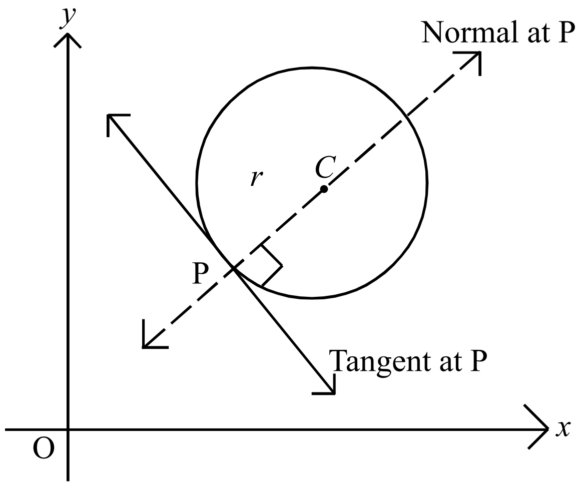
$\Rightarrow \quad \sqrt{g^2+f^2-c} = \dfrac{|l(-g)+m(-f)+n|}{\sqrt{l^2+m^2}}$ is the condition

Normal: The normal at any point P on the circle, is the line which passes through P and is perpendicular to the tangent at P.

The equation of normal at P is the equation of the line passing through two points C and P.

- The equation of the normal at $P(x_1, y_1)$ on the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is the equation of CP [$\because$ Centre $= (-g, -f) = C$, $P(x_1, y_1)$]

  i.e. $y - y_1 = \dfrac{y_1 + f}{x_1 + g}(x - x_1)$ [$\because$ (two points form) equation of CP]



- The length of the chord AB $= 2\sqrt{r^2 - d^2}$

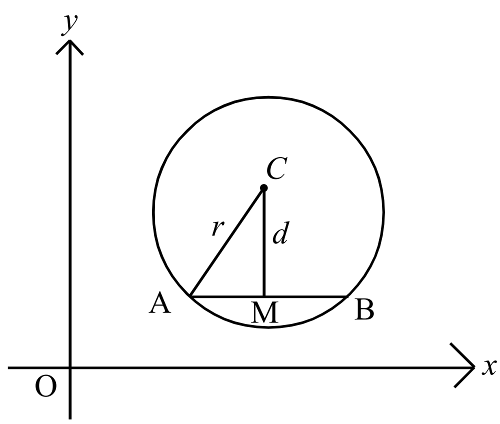
  where 'r' is the radius of the circle and 'd' is the length of the perpendicular drawn from the centre to the chord AB

  In Δ ACM, $r^2 = d^2 + (AM)^2$

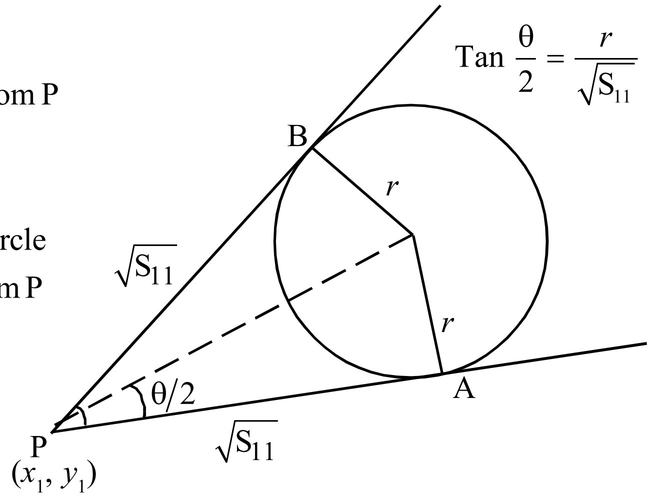
$\Rightarrow \quad (AM)^2 = r^2 - d^2$

$\Rightarrow \quad \overline{AM} = \sqrt{r^2 - d^2}$

length of chord $\overline{AB} = 2\overline{AM} = 2\sqrt{r^2 - d^2}$



Chord of Contact, Pole, Polar



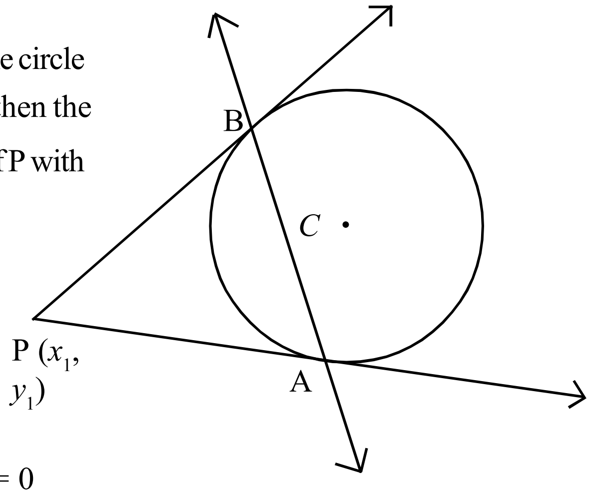
PB = PA = Length of tangent drawn from P

$= \sqrt{S_{11}}$

If $P(x_1, y_1)$ is an external point of the circle $S = 0$, then there exists two tangents from P to the circle $S = 0$.

- If θ is the angle between the tangents through an external point $P(x_1\ y_1)$ to the circle $S = 0$, then

  $\tan\left(\frac{\theta}{2}\right) = \frac{r}{\sqrt{S_{11}}}$, where r is the radius of the circle

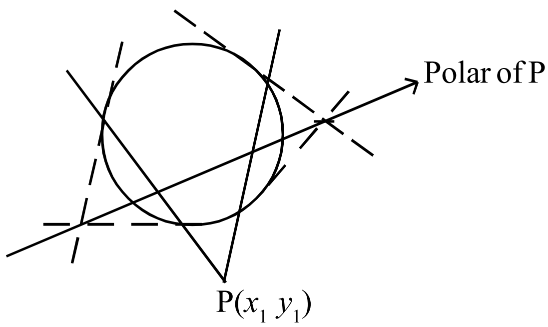


Chord of Contact of P (equation is $S_1 = 0$)

- If the tangents drawn through $P(x_1, y_1)$ to the circle $S = 0$, touch the circle at points A and B, then the secant $\overleftrightarrow{AB}$ is called the **chord of contact** of P with respect to the circle $S = 0$
- If $P(x_1, y_1)$ is an exterior point to the circle $S = 0$, then the equation of chord of contact of P with respect to the circle $S = 0$ is $S_1 = 0$ that is

  $x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c = 0$
- **pole and polar $\rightarrow$ definition equations**
- Let $S = 0$ be a circle and P be any point in the plane other than the centre of $S = 0$. Then the polar of P is the locus of the point of intersection of tangents drawn at the extremities of the chord passing through P.

  P is called as the pole of the polar.



- The equation of the polar of $P(x_1, y_1)$ with respect to the circle $S = 0$ is $S_1 = 0$

  The pole of the line $lx + my + n = 0$, $(n \neq 0)$ with respect to the circle $x^2 + y^2 = a^2$ is $\left(\frac{-a^2 l}{n}, \frac{-a^2 m}{n}\right)$

- The pole of $lx + my + n = 0$ with respect to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is

  $\left(-g + \frac{lr^2}{lg + mf - n}, -f + \frac{mr^2}{lg + mf - n}\right)$ where r is the radius of the circle.

- The polar of $P(x_1, y_1)$ w.r.t the circle $S = 0$ passes through $Q(x_2, y_2) \Leftrightarrow$ the polar of Q passes through P.

- Two points P and Q are said to be **conjugate points** with respect to the circle $S = 0$ if Q lies on the polar of P. (Then P lies on the polar of Q also)

- The condition that the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points with respect to the circle $S = 0$ is $S_{12} = 0$

  That is $x_1 x_2 + y_1 y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$

- If P and Q are conjugate points with respect to the circle $S = 0$, then the polars of P and Q are called as **conjugate lines** with respect to the circle $S = 0$

  or

  Two straight lines are said to be **conjugate lines** with respect to the circle $S = 0$, if the pole of one line, lies on the other line.

- The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate lines with respect to the circle $x^2 + y^2 = a^2$ is $a^2(l_1 l_2 + m_1 m_2) = n_1 n_2$

- The condition for the lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ to be conjugate lines with respect to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ is

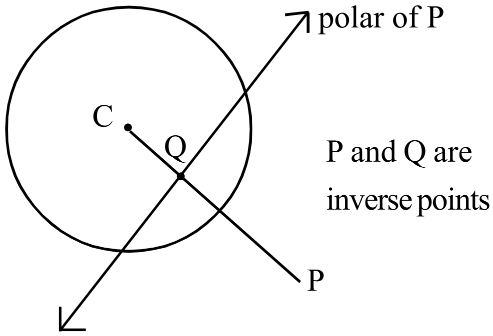
  $r^2(l_1 l_2 + m_1 m_2) = (l_1g + m_1f - n_1) \times (l_2g + m_2f - n_2)$ where $r = \sqrt{g^2 + f^2 - c}$

- Let C be the centre and 'r' be the radius of the circle $S = 0$. Two points P and Q are said to be **inverse points** with respect to the circle $S = 0$, if the points C, P, Q are collinear such that P and Q are on the same side of C and $(CP) \times (CQ) = r^2$

Theorem :

Let 'C' be the centre and 'r' be the radius of the circle S = 0.

Two points P and Q are inverse points if and only if, Q is the point of intersection of the polar of P w.r.t the circle S = 0 and the line joining P and C.



- The inverse of the point P with respect to the circle S = 0 is the foot of the perpendiuclar drawn from the centre of the circle S = 0 to the polar of P.

Problem

1. Find the inverse point of (–2, 3) with respect to the circle $x^2 + y^2 - 4x - 6y + 9 = 0$

Sol : The given circle is $S = x^2 + y^2 - 4x - 6y + 9 = 0$ ... (1)

comparing with the standard equation we get

$2g = -4 \Rightarrow g = -2$

$2f = -6 \Rightarrow f = -3$

$c = 9$

$\therefore$ centre $= (-g, -f) = (2, 3) = C$

Let P = (–2, 3)

equation of CP is $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$\Rightarrow \quad y - 3 = \dfrac{3-3}{-2-2}(x-2)$

$\Rightarrow \quad y - 3 = 0$ ... (2)

The polar of P is $S_1 = 0$ where $P(x_1, y_1) = (-2, 3)$

$\Rightarrow \quad x x_1 + y y_1 - 2(x + x_1) - 3(y + y_1) + 9 = 0$

$\Rightarrow \quad x(-2) + y(3) - 2(x - 2) - 3(y + 3) + 9 = 0$

$\Rightarrow \quad -2x + 3y - 2x + 4 - 3y - 9 + 9 = 0$

$\Rightarrow \quad -4x + 4 = 0$

$\Rightarrow \quad 4(-x + 1) = 0$

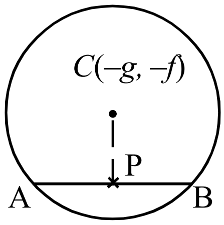
$\Rightarrow \quad -x + 1 = 0$ ... (3)

Solving (2) and (3), we get

$x = 1, y = 3$

$\therefore$ The inverse point of P is Q = (1, 3)

- If $P(x_1, y_1)$ is the midpoint of the chord $\overline{AB}$ (other than the diameter) of the circle S = 0, then the equation of secant $\overleftrightarrow{AB}$ is $S_1 = S_{11}$



$P(x_1, y_1)$ is the midpoint of the chord AB

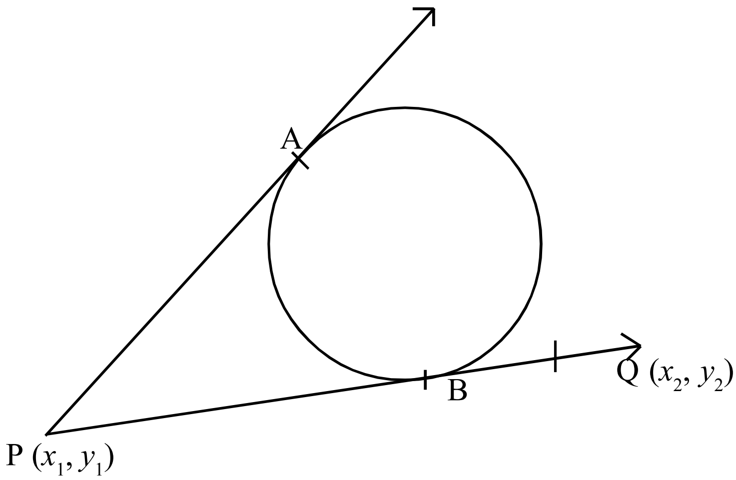
equation of $\overleftrightarrow{AB}$ is $S_1 = S_{11}$

That is, $x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$

- **Very important. (learn the derivation)**

Show that the combined equation of the pair of tangents drawn from an external point $P(x_1, y_1)$ to the circle S = 0 is $S_1^2 = S.S_{11}$.

Sol :



Let A and B be the points of contact of tangents drawn from $P(x_1, y_1)$ to the circle S = 0

Then $\overleftrightarrow{AB}$ is the chord of contact of P and its equation is $S_1 = 0$.

i.e., $x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c = 0$

Let $Q(x_2, y_2)$ be any point on one of the tangents

Now the locus of Q is the equation of the pair of tangents drawn from P.

The line $\overleftrightarrow{AB}$ ie, $S_1 = 0$ divides $\overleftrightarrow{PQ}$ in the ratio $-\dfrac{S_{11}}{S_{12}}$

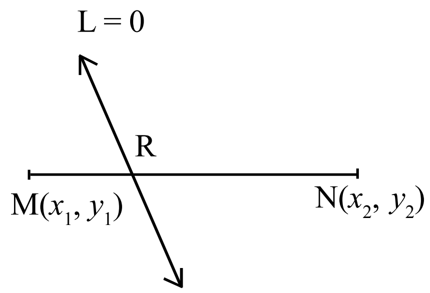
$\Rightarrow \quad \dfrac{PB}{BQ} = \dfrac{-S_{11}}{S_{12}} \qquad \ldots (1)$

But PB = $\sqrt{S_{11}}$ = length of tangent drawn from P

BQ = $\sqrt{S_{22}}$ = Length of tangent drawn from Q

$\therefore \quad \dfrac{PB}{BQ} = \dfrac{\sqrt{S_{11}}}{\sqrt{S_{22}}} \qquad \ldots (2)$

We know that



R divides $\overline{MN}$ in the ratio $-\dfrac{L_{11}}{L_{22}}$

From (1) & (2), we get $\dfrac{\sqrt{S_{11}}}{\sqrt{S_{22}}} = -\dfrac{S_{11}}{S_{12}}$

Squaring on both sides, we get $\dfrac{S_{11}}{S_{22}} = \dfrac{S_{11}^{\ 2}}{S_{12}^{\ 2}} \Rightarrow \dfrac{1}{S_{22}} = \dfrac{S_{11}}{S_{12}^{\ 2}}$

$\Rightarrow \quad S_{12}^{\ 2} = S_{11} \cdot S_{22}$

$\therefore$ The locus of $Q(x_2, y_2)$ is

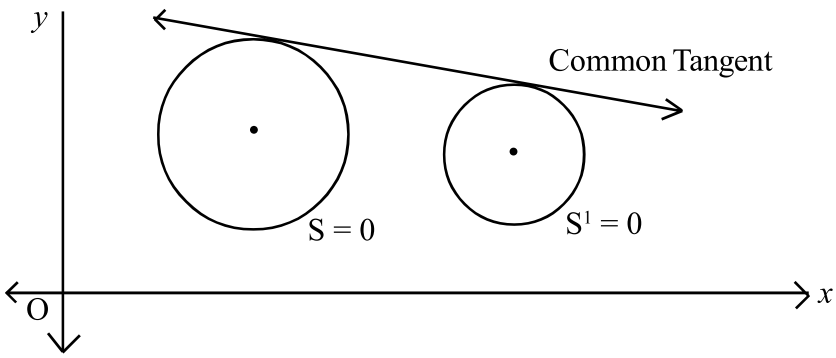
$S_1^{\ 2} = S_{11} \cdot S$

$\Rightarrow \quad S_1^{\ 2} = S \cdot S_{11}$ is the equation of the pair of tangents drawn from an external point $P(x_1, y_1)$ to the circle S = 0.

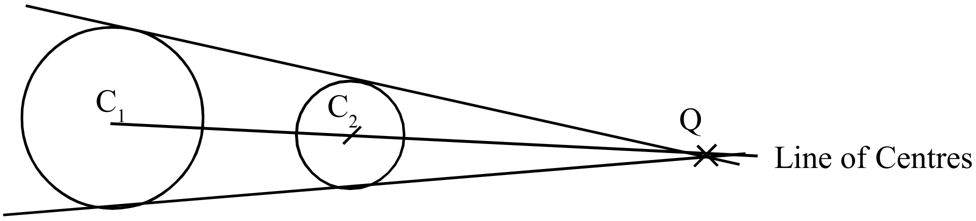
Hence proved.

Common Tangents

- A straight line L is said to be a common tangent to the circles S = 0 and $S^1 = 0$, if it is a tangent to both S = 0 and $S^1 = 0$.



- Any two intersecting common tangents of two circles and the line joining the centres of the circles are concurrent.

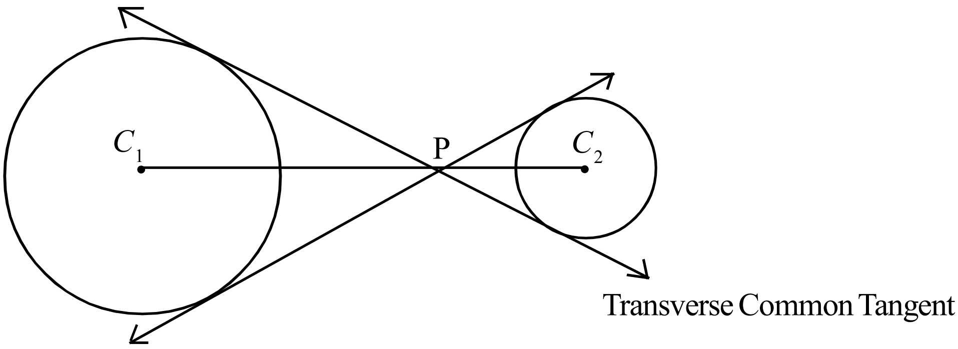


The two common tangents and the line of centres intersect at Q (concurrent at Q)

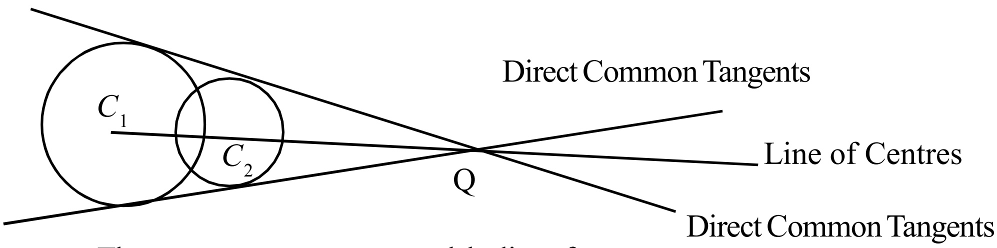
- The point of intersection Q, of two common tangents (if exists) of two circles and the centres C_1 and C_2 of these two circles are collinear.

  C_1, C_2, Q are collinear (lie on a straight line)

- The pair of common tangents to the circles S = 0 and $S^1 = 0$, touching at a point on the line segment $\overline{C_1C_2}$ (C_1, C_2 are the centres of the circles) is called **transverse pair of common tangents**.



- The pair of common tangents to the circles S = 0 and $S^1 = 0$, intersecting at a point not in $\overline{C_1C_2}$ is called as **direct pair of common tangents**



The two common tangents and the line of centres intersect at Q (concurrent at Q)

- The point of intersection P, of transverse pair of common tangents is called as **Internal centre of similitude.**

- The point P, divides the segment $\overline{C_1C_2}$ in the ratio $r_1 : r_2$ internally. (where r_1 is the radius of the circle with centre C_1 and r_2 is the radius of the circle with centre C_2)
- The point of intersection Q, of direct pair of common tangents is called as **external centre of similitude.**
- The point Q, divides the segment $\overline{C_1C_2}$ in the ratio $r_1 : r_2$ externally.
- P, Q, C_1, C_2 are all collinear.

Where P is the internal centre of similitude,

Q is the external centre of similitude,

C_1, C_2 are the centres of the two circles.

Relative positions of two circles

Let C_1, C_2 be the centres and r_1, r_2 be the radii of two circles S = 0, S^1 = 0 respectively.

Let $\overline{C_1C_2}$ represent the line segment from C_1 to C_2.

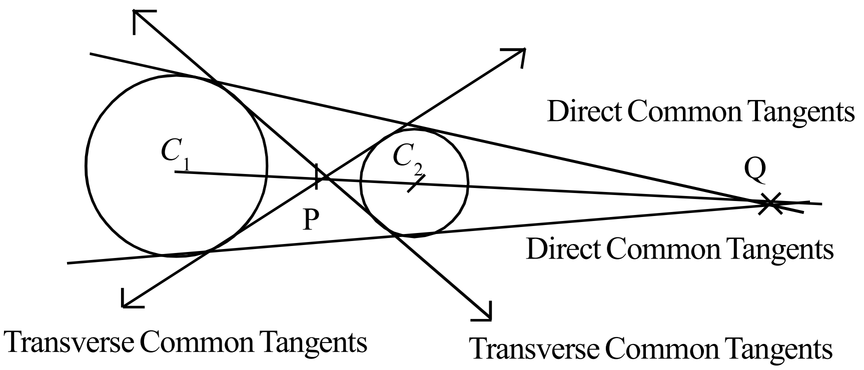
The following cases arise with regard to the relative position of two circles.

Case (i)

each of the given pair of circles lies in the exterior of the other

condition : $\overline{C_1C_2} > r_1 + r_2, (r_1 \neq r_2)$

In this case the two circles do not intersect.



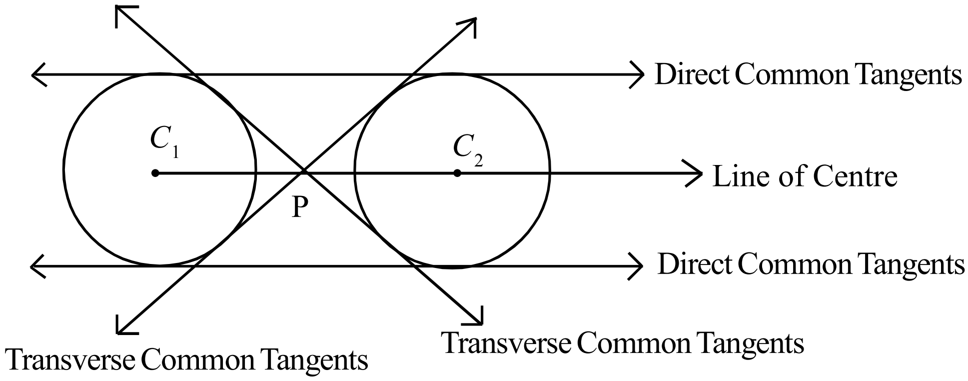
For two non - intersecting circles, we can draw two direct common tangents and two transverse common tangents

So we can draw FOUR COMMON TANGENTS

P is the internal centre of similitude, Q is the external centre of similitude

Case (ii)

Condition: $\overline{C_1C_2} > r_1 + r_2, r_1 = r_2$



The circles are non - intersecting circles.

The transverse common tangents intersect at P, the internal centre of similitude.

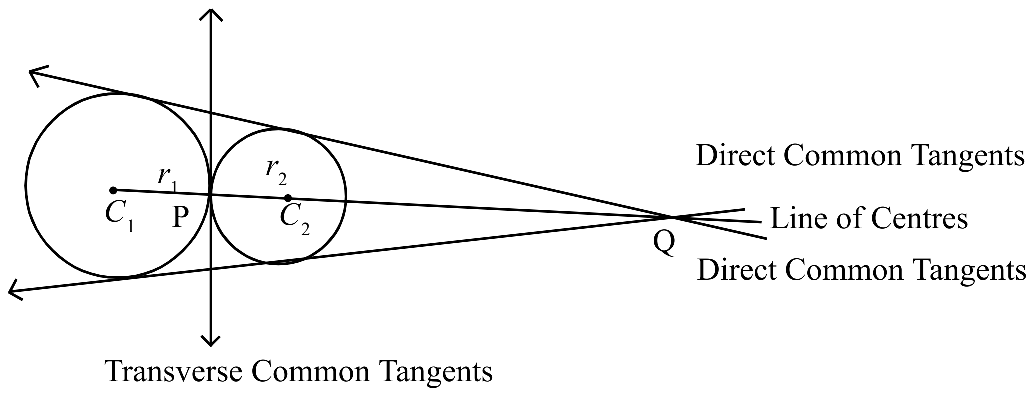
The direct common tangents are parallel to $\overline{C_1C_2}$

The external centre of similitude, Q, does not exist.

So we can draw four common tangents

Case (iii)

Condition : $\overline{C_1C_2} = r_1 + r_2$



The two circles touch each other externally.

The internal centre of similitude ‘P’ is the point of contact of the two given circles

At ‘P’, there is only one transverse common tangents.

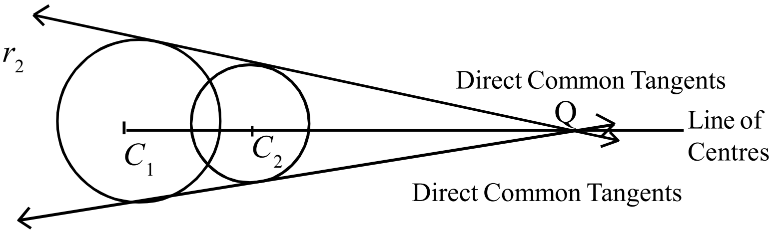
The direct common tangents intersect at Q, the external centre of similitude.

So in this case we can draw THREE COMMON TANGENTS

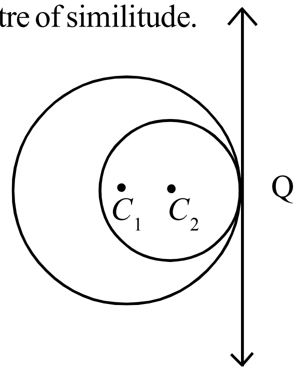
Case (iv)

Condition : $|r_1 - r_2| < \overline{C_1C_2} < r_1 + r_2$

In this case the two circles intersect each other.



In this case the two direct common tangents intersect at Q, the external centre of similitude.
We cannot draw transverse common tangents
So the internal centre of similitude does not exist.
In this case we can draw only Two common tangents



Case (v)

Condition : $\overline{C_1C_2} = |r_1 - r_2|$

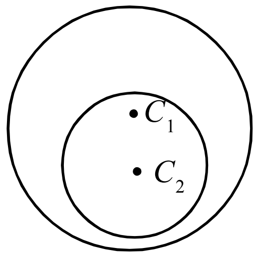
In this case, the two circles touch each other internally
we cannot draw transverse common tangents

$\Rightarrow$ The internal centre of similitude does not exist
Only one direct common tangent can be drawn at the point of contact, Q, of the two circles.
In this case, we can draw only ONE COMMON TANGENT

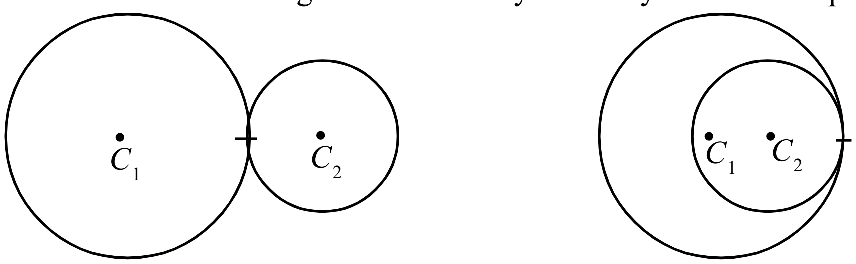
Case : (vi)

Condition : $\overline{C_1C_2} < |r_1 - r_2|$

In this case one circle lies entirely in the interior of the other circle.
The number of common tangents that can be drawn to the two circles is zero
No.of common tangents = zero



Note : Two circles are said to be touching each other if they have only one common point

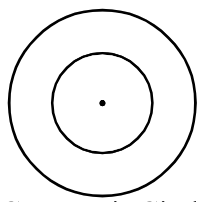


Case : (vii)

$\Rightarrow$ If $C_1C_2 = 0$, then the centres of the two circles, coincide

$\Rightarrow$ They are concentric circles

$\Rightarrow$ The no. of common tangents drawn to the two circles is zero


Concentric Circles

Problems

1. Find the equation of the circle whose centre is (2, 3) and radius is 5.

Sol : Equation of the circle whose centre is $(a, b) = (2, 3)$ and radius, $r = 5$ is

$(x - a)^2 + (y - b)^2 = r^2$

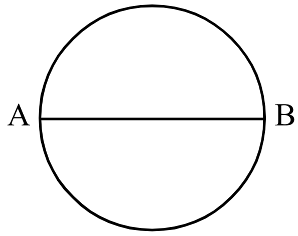
$\Rightarrow$ $(x - 2)^2 + (y - 3)^2 = 5^2$

$\Rightarrow$ $x^2 + 4 - 4x + y^2 + 9 - 6y - 25 = 0$

$\Rightarrow$ $x^2 + y^2 - 4x - 6y - 12 = 0$

2. If the extremities of diameter of a circle are (3, 5) and (9,3), then find the equation of the circle.

Sol : The equation of the circle whose ends of the diameter



are $A = (x_1, y_1) = (3, 5)$ and $B = (x_2, y_2) = (9, 3)$

is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\Rightarrow$ $(x - 3)(x - 9) + (y - 5)(y - 3) = 0$

$\Rightarrow$ $x^2 - 9x - 3x + 27 + y^2 - 3y - 5y + 15 = 0$

$\Rightarrow$ $x^2 + y^2 - 12x - 8y + 42 = 0$

3. Find the centre and radius of each of the following circles.

(i) $x^2 + y^2 - 4x - 8y - 41 = 0$

(ii) $3x^2 + 3y^2 - 5x - 6y + 4 = 0$

Sol : (i) Given circle is $x^2 + y^2 - 4x - 8y - 41 = 0$

Comparing it with the standard equation of the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = -4,\ 2f = -8,\ c = -41$

$\Rightarrow \quad g = \dfrac{-4}{2} = -2,\ f = \dfrac{-8}{2} = -4, c = -41.$

$\therefore \quad$ centre $= (-g, -f) = (-(-2), -(-4)) = (2, 4)$

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-4)^2 - (-41)} = \sqrt{4+16+41} = \sqrt{61}$

(ii) Given circle is $3x^2 + 3y^2 - 5x - 6y + 4 = 0$

$\Rightarrow \quad \dfrac{3x^2}{3} + \dfrac{3y^2}{3} - \dfrac{5x}{3} - \dfrac{6y}{3} + \dfrac{4}{3} = 0$

$\Rightarrow \quad x^2 + y^2 - \dfrac{5}{3}x - 2y + \dfrac{4}{3} = 0$

Comparing this equation with the standard equation

$x^2 + y^2 + 2gx + 2fy + c = 0$ of the circle, we get

[Note : Always write the equation of the circle in the standard form with coefficient of x^2 and y^2 as one. So divide all the terms by 3, so that coefficient of x^2 & y^2 becomes one]

$2g = \dfrac{-5}{3},\ 2f = -2,\ c = \dfrac{4}{3}$

$\Rightarrow \quad g = \dfrac{-5}{6},\ f = -1,\ c = \dfrac{4}{3}$

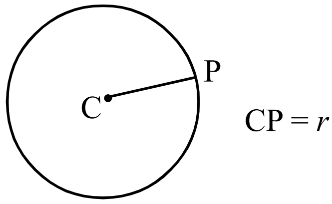
$\therefore \quad$ centre $= (-g, -f) = \left(\dfrac{5}{6}, 1\right)$

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{\left(\dfrac{-5}{6}\right)^2 + (-1)^2 - \dfrac{4}{3}}$

$= \sqrt{\dfrac{25}{36} + 1 - \dfrac{4}{3}} = \sqrt{\dfrac{25+36-48}{36}}$

$= \sqrt{\dfrac{13}{36}} = \dfrac{\sqrt{13}}{\sqrt{36}} = \dfrac{\sqrt{13}}{6}$

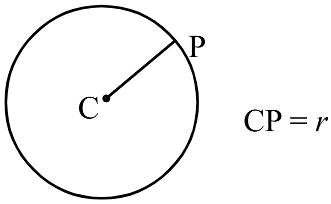
Note : When C is the centre of the circle, and if the circle passes through the point P, the distance CP is the radius of the circle.



4. Find the equation of the circle passing through the point (2, –1) and having centre at (2, 3)

Sol : Let P = (2,–1)

Centre = C = (a, b) = (2, 3)



Since the circle passes through the point P,

radius = distance CP

$$= \sqrt{(2-2)^2 + (3+1)^2} \qquad \left(\because \text{distance formula} : \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right)$$

$$= \sqrt{0+16}$$

$$= \sqrt{16} = 4 = r.$$

$\therefore$ The equation of the required circle is $(x - a)^2 + (y - b)^2 = r^2$

$\Rightarrow$ $(x - 2)^2 + (y - 3)^2 = 4^2$

$\Rightarrow$ $x^2 + y^2 - 4x - 6y + 4 + 9 - 16 = 0$

$\Rightarrow$ $x^2 + y^2 - 4x - 6y - 3 = 0$

Second Method

Let the equation of the circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1)

Its centre is $(-g, -f) = (2, 3)$

$\Rightarrow$ $-g = 2, -f = 3$

$\Rightarrow$ $\boxed{g = -2}, \boxed{f = -3}$

Now the circle (1) becomes $x^2 + y^2 + 2(-2)x + 2(-3)y + c = 0$

$\Rightarrow$ $x^2 + y^2 - 4x - 6y + c = 0$ ... (2)

It passes through the point (2, –1)

$\Rightarrow$ It satisfies the point (2, –1). Substituting in (2), we get

$(2)^2 + (-1)^2 - 4(2) - 6(-1) + c = 0$

$\Rightarrow$ $4 + 1 - 8 + 6 + c = 0$

$\Rightarrow$ $\boxed{c = -3}$

Substituting the values of g, f, c in (1) we get the required circle as

$$x^2 + y^2 - 4x - 6y - 3 = 0$$

5. Obtain the parametric equations of the following circles.

(i) $4(x^2 + y^2) = 9$

(ii) $x^2 + y^2 - 4x - 6y - 12 = 0$

Sol :

(i) Given circle is $4(x^2 + y^2) = 9 \quad \Rightarrow \quad x^2 + y^2 = \frac{9}{4}$

Comparing this equation with $x^2 + y^2 = r^2$ we get $r^2 = \frac{9}{4}$

centre of the circle is $(0, 0) = (x_1, y_1) \quad \Rightarrow \quad r = \sqrt{\frac{9}{4}} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2}$

$\therefore$ The parametric equations of the circle are

$$\left.\begin{aligned} x &= x_1 + r\cos\theta \\ y &= y_1 + r\sin\theta \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= 0 + \frac{3}{2}\cos\theta \\ y &= 0 + \frac{3}{2}\sin\theta \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= \frac{3}{2}\cos\theta \\ y &= \frac{3}{2}\sin\theta \end{aligned}\right\}$$

where (x_1, y_1) = centre, $0 \le \theta < 2\pi$

(ii) Given circle is $x^2 + y^2 - 4x - 6y - 12 = 0$

Comparing with the standard equation $x^2 + y^2 + 2gx + 2fy + c = 0$

we get $2g = -4, \quad 2f = -6, \ c = -12$

$g = -2, \quad f = -3, \ c = -12$

$\therefore$ centre $= (-g, -f) = (2, 3) = (x_1, y_1)$

radius $= r = \sqrt{g^2 + f^2 - c} = \sqrt{4 + 9 + 12} = \sqrt{25} = 5$

$\therefore$ The parametric equations of the circle are

$$\left.\begin{aligned} x &= x_1 + r\cos\theta \\ y &= y_1 + r\sin\theta \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x &= 2 + 5\cos\theta \\ y &= 3 + 5\sin\theta, \ 0 \le \theta < 2\pi \end{aligned}\right\}$$

6. Find the values of a, b if $ax^2 + bxy + 3y^2 - 5x + 2y - 3 = 0$

represents a circle. Also find the radius and centre of the circle

Sol : The given equation is $ax^2 + bxy + 3y^2 - 5x + 2y - 3 = 0$

It represents a circle if coefficient of x^2 = coefficient of y^2

and coefficient of xy is zero

$\Rightarrow$ $a = 3$ and $b = 0$

$\therefore$ The circle is $3x^2 + 3y^2 - 5x + 2y - 3 = 0$

Divide by 3,

$$\Rightarrow \quad \frac{3x^2}{3} + \frac{3y^2}{3} - \frac{5x}{3} + \frac{2y}{3} - \frac{3}{3} = \frac{0}{3}$$

$$\Rightarrow \quad x^2 + y^2 - \frac{5}{3}x + \frac{2}{3}y - 1 = 0$$

comparing this equation with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$$2g = \frac{-5}{3}, \ \ 2f = \frac{2}{3}, \ \ c = -1$$

$$\Rightarrow \quad g = \frac{-5}{6}, \ f = \frac{1}{3}, \ c = -1$$

$\therefore$ centre $= (-g, -f) = \left(\frac{5}{6}, -\frac{1}{3}\right)$

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{5}{6}\right)^2 + \left(-\frac{1}{3}\right)^2 - (-1)}$

$$= \sqrt{\frac{25}{36} + \frac{1}{9} + 1} = \sqrt{\frac{25 + 4 + 36}{36}} = \sqrt{\frac{65}{36}} = \frac{\sqrt{65}}{\sqrt{36}} = \frac{\sqrt{65}}{6}$$

$\therefore$ radius $= \frac{\sqrt{65}}{6}$

7. If $x^2 + y^2 - 4x + 6y + c = 0$ represents a circle with radius 6, then find c.

Sol. Comparing the given circle $x^2 + y^2 - 4x + 6y + c = 0$ with

the standard equation $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = -4,\ 2f = 6,\ c = c$

$\Rightarrow \quad g = -2, f = 3, c = c$

$\therefore \quad$ radius = 6 $\quad \Rightarrow \quad \sqrt{g^2 + f^2 - c} = 6$

Squaring on both sides, we get

$g^2 + f^2 - c = 6^2$

$\Rightarrow \quad (-2)^2 + 3^2 - c = 36$

$\Rightarrow \quad -c = 36 - 4 - 9$

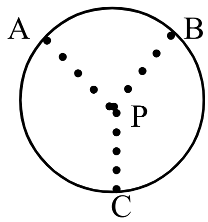
$\Rightarrow \quad -c = 23 \quad \Rightarrow \quad \boxed{c = -23}$

8. Find the equation of the circle passing through the three points (3, –4), (1, 2), (5, –6)

Sol : Let A = (3, –4), B = (1, 2), C = (5, –6)

let $P(x_1, y_1)$ be the centre of the circle passing through the points A, B and C

Then PA = PB = PC = radius of the circle



Now PA = PB $\Rightarrow \quad \sqrt{(x_1-3)^2 + (y_1+4)^2} = \sqrt{(x_1-1)^2 + (y_1-2)^2}$

Squaring on both sides we get, $(x_1 - 3)^2 + (y_1 + 4)^2 = (x_1 - 1)^2 + (y_1 - 2)^2$

$\Rightarrow \quad x_1^2 - 6x_1 + 9 + y_1^2 + 8y_1 + 16 = x_1^2 - 2x_1 + 1 + y_1^2 - 4y_1 + 4$

$\Rightarrow \quad -6x_1 + 8y_1 + 25 + 2x_1 + 4y_1 - 5 = 0$

$\Rightarrow \quad -4x_1 + 12y_1 + 20 = 0$

$\Rightarrow \quad 4(-x_1 + 3y_1 + 5) = 0$

$\Rightarrow \quad -x_1 + 3y_1 + 5 = 0 \qquad \ldots (1)$

Again PB = PC $\quad \Rightarrow \quad$ Squaring on both sides, $PB^2 = PC^2$

$\Rightarrow \quad (x_1 - 1)^2 + (y_1 - 2)^2 = (x_1 - 5)^2 + (y_1 + 6)^2$

$\Rightarrow \quad x_1^2 - 2x_1 + 1 + y_1^2 - 4y_1 + 4 = x_1^2 - 10x_1 + 25 + y_1^2 + 12y_1 + 36$

$\Rightarrow \quad -2x_1 - 4y_1 + 5 + 10x_1 - 12y_1 - 61 = 0$

$\Rightarrow \quad 8x_1 - 16y_1 - 56 = 0$

$\Rightarrow \quad 8(x_1 - 2y_1 - 7) = 0$

$\Rightarrow \quad x_1 - 2y_1 - 7 = 0 \qquad \ldots (2)$

Solving (1) & (2) weget

$$\begin{array}{r} -x_1 + 3y_1 + 5 = 0 \\ x_1 - 2y_1 - 7 = 0 \\ \hline y_1 - 2 = 0 \end{array}$$

$\Rightarrow \quad y_1 = 2$

substituting $y_1 = 2$ in (2), we get $x_1 - 2(2) - 7 = 0$

$\Rightarrow \quad x_1 = 4 + 7 = 11$

$\therefore \quad$ P $= (x_1, y_1) = (11, 2)$ is the centre of the circle

Radius = PA = $\sqrt{(11-3)^2 + (2+4)^2} = \sqrt{8^2 + 6^2} = \sqrt{64+36} = \sqrt{100} = 10$

$r = 10$

$\therefore \quad$ The equation of the circle passing through the points A, B & C

is $(x - x_1)^2 + (y - y_1)^2 = r^2$

$\Rightarrow \quad (x - 11)^2 + (y - 2)^2 = 10^2$

$\Rightarrow \quad x^2 + y^2 - 22x - 4y + 25 = 0$

Note : Centre $(a, b) = (x_1, y_1)$ and $(x - a)^2 + (y - b)^2 = r^2$ is the equation of the circle.

9. Show that the points (1, 2), (3, –4), (5, –6), (19, 8) are concyclic and find theequation of the circle on which they lie.

Sol : Let A = (1, 2), B = (3, –4), C = (5, –6), D = (19, 8) be the given points. They are concyclic, if they all lie on the same circle.

Let S $= (x_1, y_1)$ be the centre of the circle passing through the points A, B, and C.

Then, SA = SB = SC

Now SA = SB $\quad \Rightarrow \quad SA^2 = SB^2$

$\Rightarrow \quad (x_1 - 1)^2 + (y_1 - 2)^2 = (x_1 - 3)^2 + (y_1 + 4)^2$

$\Rightarrow \quad x_1^2 - 2x_1 + 1 + y_1^2 - 4y_1 + 4 = x_1^2 - 6x_1 + 9 + y_1^2 + 8y_1 + 16$

$\Rightarrow \quad -2x_1 - 4y_1 + 5 + 6x_1 - 8y_1 - 25 = 0$

$\Rightarrow \quad 4x_1 - 12y_1 - 20 = 0$

$\Rightarrow \quad 4(x_1 - 3y_1 - 5) = 0$

$\Rightarrow \quad x_1 - 3y_1 - 5 = 0 \qquad \text{... (1)}$

Again SB = SC $\quad \Rightarrow \quad SB^2 = SC^2$

$\Rightarrow \quad (x_1 - 3)^2 + (y_1 + 4)^2 = (x_1 - 5)^2 + (y_1 + 6)^2$

$\Rightarrow \quad x_1^2 - 6x_1 + 9 + y_1^2 + 8y_1 + 16 = x_1^2 - 10x_1 + 25 + y_1^2 + 12y_1 + 36$

$\Rightarrow \quad -6x_1 + 8y_1 + 25 + 10x_1 - 12y_1 - 61 = 0$

$\Rightarrow \quad 4x_1 - 4y_1 - 36 = 0$

$\Rightarrow \quad 4(x_1 - y_1 - 9) = 0$

$\Rightarrow \quad x_1 - y_1 - 9 = 0 \qquad \text{... (2)}$

Solving (1) and (2) we get

$$\begin{array}{r} x_1 - 3y_1 - 5 = 0 \\ x_1 - \ \ y_1 - 9 = 0 \\ - \quad + \quad + \quad \\ \hline -2y_1 + 4 = 0 \end{array}$$

$\Rightarrow \quad -2y_1 = -4$

$\Rightarrow \quad y_1 = \dfrac{-4}{-2} = 2$

substituting $y_1 = 2$ in (1), we get $x_1 - 3(2) - 5 = 0 \quad \Rightarrow \quad x_1 = 11$

$\therefore \quad$ centre $= (x_1, y_1) = (11, 2)$

$\therefore \quad$ radius $= SA = \sqrt{(11-1)^2 + (2-2)^2} = \sqrt{10^2 + 0^2} = \sqrt{100} = 10$

$\therefore \quad$ The equation of the circle passing through the points A, B and C is $(x - x_1)^2 + (y - y_1)^2 = r^2$

$\Rightarrow \quad (x - 11)^2 + (y - 2)^2 = 10^2$

$\Rightarrow \quad x^2 + y^2 - 22x - 4y + 121 + 4 - 100 = 0$

$\Rightarrow \quad x^2 + y^2 - 22x - 4y + 25 = 0 \qquad \text{... (3)}$

Now substituting D = (19, 8) in (3), we get

$(19)^2 + (8)^2 - 22(19) - 4(8) + 25$

$= 361 + 64 - 418 - 32 + 25$

$= 450 - 450 = 0$

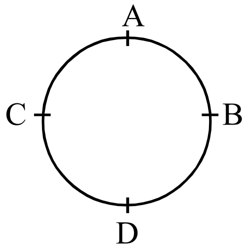
$\Rightarrow$ D lies on the circle (3), Hence proved

$\therefore$ The four points A, B, C, D lie on the circle (3)

i.e $x^2 + y^2 - 22x - 4y + 25 = 0$

The points A, B, C, D are concyclic.

Note : Four points are said to be concyclic if they all lie on the same circle.



The **equation** of the circle with centre (a, b) and radius 'r' is $(x-a)^2 + (y-b)^2 = r^2$

If centre is (x_1, y_1), Then the circle is $(x - x_1)^2 + (y - y_1)^2 = r^2$

10. If (2, 0), (0, 1), (4, 5) and (0, c) are concyclic, then find 'c'.

Sol : Let A (2, 0), B = (0, 1), C = (4, 5), D = (0, c) be the points which are concylic i.e., the points lying on the same circle.

Let S = (x_1, y_1) be the centre of the circle passing through the points A, B, C and D.

Then, SA = SB = SC = SD

Now, SA = SB. Squaring on both sides, $SA^2 = SB^2$

$\Rightarrow$ $(x_1 - 2)^2 + (y_1 - 0)^2 = (x_1 - 0)^2 + (y_1 - 1)^2$

$\Rightarrow$ $x_1^2 - 4x_1 + 4 + y_1^2 = x_1^2 + y_1^2 - 2y_1 + 1$

$\Rightarrow$ $-4x_1 + 2y_1 + 3 = 0$ ... (1)

Again, SB = SC

Squaring on both sides

$SB^2 = SC^2$

$$(x_1-0)^2+(y_1-1)^2=\left(x_1-4\right)^2+(y_1-5)^2$$

$$x_1^2+y_1^2-2y_1+1=x_1^2-8x_1+16+y_1^2-10y_1+25$$

$$8x_1+8y_1-40=0$$

$$2(4x_1+4y_1-20)=0$$

$$4x_1+4y_1-20=0 \qquad \text{.... (2)}$$

solving (1) and (2)

$$\begin{array}{r} -4x_1+2y_1+3=0 \\ 4x_1+4y_1-20=0 \\ \hline 6y_1-17=0 \end{array}$$

$\Rightarrow \quad y_1=\dfrac{17}{6}$

substituing $y_1=\dfrac{17}{6}$ in (1) we get

$$-4x_1+2\left(\frac{17}{6}\right)+3=0$$

$\Rightarrow \quad -4x_1+\dfrac{17}{3}+3=0$

$\Rightarrow \quad -4x_1+\dfrac{17+9}{3}=0$

$\Rightarrow \quad -4x_1=-\dfrac{26}{3}$

$\Rightarrow \quad x_1=\dfrac{-26}{-12}=\dfrac{13}{6}$

$\therefore \quad \left(x_1,y_1\right)=\left(\dfrac{13}{6},\dfrac{17}{6}\right)$

Now, SC = SD $\Rightarrow$ Squaring on both sides, $SC^2 = SD^2$ or SA = SD $\Rightarrow$ $SA^2 = SD^2$

Here we can take SA = SD (or) SB = SC (or) SC = SD

Since SA is simple as A = (2, 0), taking SA = SD

Squaring on both sides, $SA^2 = SD^2$

$\Rightarrow \quad (x_1 - 2)^2 + (y_1 - 0)^2 = (x_1 - 0)^2 + (y_1 - c)^2$

$\Rightarrow \quad x_1^2 - 4x_1 + 4 + y_1^2 = x_1^2 + y_1^2 - 2cy_1 + c^2$

$\Rightarrow \quad -4x_1 + 4 = -2cy_1 + c^2$

$$\Rightarrow \quad -4\left(\frac{13}{6}\right) + 4 = -2c\left(\frac{17}{6}\right) + c^2$$

$$\Rightarrow \quad \frac{-26}{3} + 4 = \frac{-17c}{3} + c^2$$

$$\Rightarrow \quad \frac{-26+12}{3} = \frac{-17c+3c^2}{3}$$

$\Rightarrow \quad -14 = -17c + 3c^2$

$\Rightarrow \quad 3c^2 - 17c + 14 = 0$

$\Rightarrow \quad (c - 1)(3c - 14) = 0$

or $$c = \frac{17 \pm \sqrt{289-168}}{2(3)} = \frac{17 \pm \sqrt{121}}{6} = \frac{17 \pm 11}{6}$$

$$= \frac{28}{6} \text{ or } \frac{6}{6}$$

$$= \frac{14}{3} \text{ or } 1$$

$\therefore \quad c = 1$ or $\frac{14}{3}$, But when $c = 1$, the point D is (0, 1) which is same as point B. Since A, B, C, D are four different points, $D = (0, c) = \left(0, \frac{14}{3}\right)$

$$\Rightarrow \quad \boxed{c = \frac{14}{3}}$$

11. Find the equation of the circle passing through (2, 3) and concentric with the circle $x^2 + y^2 + 8x + 12y + 15 = 0$

Sol : Given circle is $x^2 + y^2 + 8x + 12y + 15 = 0$ ... (1)

The equation of any circle concentric with (1) is

$x^2 + y^2 + 8x + 12y + k = 0$ ... (2)

($\because$ centres of concentric circles are same)

It passes through the point (2, 3)

$\Rightarrow$ $2^2 + 3^2 + 8(2) + 12(3) + k = 0$

$\Rightarrow$ $4 + 9 + 16 + 36 + k = 0$

$\Rightarrow$ $k = -65$

substituting $k = -65$ in (2), we get the required circles as

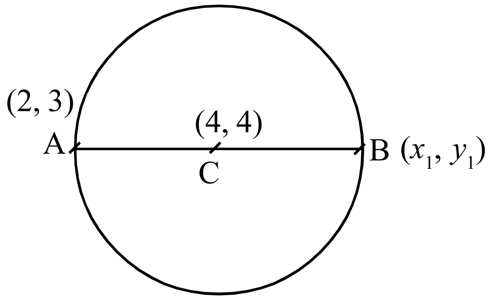
$x^2 + y^2 + 8x + 12y - 65 = 0$ **Ans**

12. Show that A(2, 3) lies on the circle $x^2 + y^2 - 8x - 8y + 27 = 0$

Also find the other end of the diameter through A

Sol : Given circle is $x^2 + y^2 - 8x - 8y + 27 = 0$ ... (1)

Substituting A(2, 3) in it we get



$2^2 + 3^2 - 8(2) - 8(3) + 27 = 0$

$= 4 + 9 - 16 - 24 + 27$

$= 40 - 40 = 0$

$\Rightarrow$ A lies on the circle (1)

Let C be the centre and AB be the diameter of the circle

$x^2 + y^2 - 8x - 8y + 27 = 0$

comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$, we get

$2g = -8$ $\quad$ $2f = -8$ $\quad$ $c = 27$

$\Rightarrow g = -4$ $\quad$ $\Rightarrow f = -4$

Centre = C = $(-g, -f) = (4, 4)$

Let A = (2, 3) and B = (x_1, y_1) be the other end ofthe diameter AB.

Then C is the midpoint of A, B.

$$\Rightarrow \quad (4, 4) = \left(\frac{2 + x_1}{2}, \frac{3 + y_1}{2}\right)$$

$$\Rightarrow \quad 4 = \frac{2 + x_1}{2}, \quad 4 = \frac{3 + y_1}{2}$$

$\Rightarrow \quad 8 = 2 + x_1, \qquad 8 = 3 + y_1,$

$\Rightarrow \quad x_1 = 6, \qquad y_1 = 5$

$\Rightarrow \quad B = (x_1, y_1) = (6, 5)$ is the other end of the diameter

13. Find the equation of the circle passing through (4, 1), (6, 5) and having the centre on the line $4x + y - 16 = 0$

Sol : First Method

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1)

be the circle passing through the points A (4, 1) & B (6, 5)

Then A lies on (1)

$\Rightarrow \quad 4^2 + 1^2 + 2g(4) + 2f(1) + c = 0$

$\Rightarrow \quad 17 + 8g + 2f + c = 0$ ... (2)

Again B(6, 5) lies on (1)

$\Rightarrow \quad 6^2 + 5^2 + 2g(6) + 2f(5) + c = 0$

$\Rightarrow \quad 61 + 12g + 10f + c = 0$ ... (3)

Now centre $(-g, -f)$ lies on $4x + y - 16 = 0$

$\Rightarrow \quad 4(-g) + (-f) - 16 = 0$

$\Rightarrow \quad -(4g + f + 16) = 0$

$\Rightarrow \quad 4g + f + 16 = 0$ ... (4)

(2) – (3)

$$\Rightarrow \quad \begin{array}{r} 17 + 8g + 2f + c = 0 \\ 61 + 12g + 10f + c = 0 \\ - \quad - \quad - \quad - \\ \hline -44 - 4g - 8f = 0 \end{array}$$

$\Rightarrow \quad -4(11 + g + 2f) = 0$

$\Rightarrow \quad 11 + g + 2f = 0$ ... (5)

Solving (4) & (5) we get

$$
\begin{array}{ll}
2 \times (4) \Rightarrow & 8g + 2f + 32 = 0 \\
& \;\; g + 2f + 11 = 0 \\
& - \quad - \quad - \\ \hline
& 7g + \quad 21 = 0
\end{array}
$$

$\Rightarrow \quad g = \dfrac{-21}{7} = -3 \qquad \Rightarrow \quad \boxed{g = -3}$

Substituting $g = -3$ in (4), we get

$4(-3) + f + 16 = 0$

$\Rightarrow \quad f = -16 + 12 = -4 \qquad \Rightarrow \quad \boxed{f = -4}$

Substituting $g = -3, f = -4$ in (3), we get

$61 + 12(-3) + 10(-4) + c = 0$

$\Rightarrow \quad 61 - 36 - 40 + c = 0$

$\Rightarrow \quad \boxed{c = 15}$

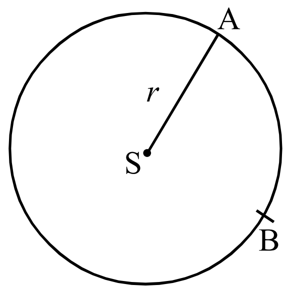
Substituting the values of g, f, c in (1) we get the required circle

$x^2 + y^2 + 2(-3)x + 2(-4)y + 15 = 0$

$\Rightarrow \quad x^2 + y^2 - 6x - 8y + 15 = 0$

Second Method

Let $S = (x_1, y_1)$ be the centre of the circle passing through the points A(4, 1) & B(6, 5). Then SA = SB



$\Rightarrow \quad SA^2 = SB^2$

$\Rightarrow \quad (x_1 - 4)^2 + (y_1 - 1)^2 = (x_1 - 6)^2 + (y_1 - 5)^2$

$\Rightarrow \quad x_1^2 - 8x_1 + 16 + y_1^2 - 2y_1 + 1$

$\qquad = x_1^2 - 12x_1 + 36 + y_1^2 - 10y_1 + 25$

$\Rightarrow \quad -8x_1 - 2y_1 + 17 + 12x_1 + 10y_1 - 61 = 0$

$\Rightarrow \quad 4x_1 + 8y_1 - 44 = 0 \qquad \ldots (1)$

Now the centre $S(x_1, y_1)$ lies on $4x + y - 16 = 0$

$\Rightarrow \quad 4x_1 + y_1 - 16 = 0 \qquad \ldots (2)$

Solving (1) & (2), we get

$$4x_1 + 8y_1 + 44 = 0$$
$$4x_1 + y_1 - 16 = 0$$
$$-\quad - \qquad +$$
$$7y_1 - 28 = 0 \quad \Rightarrow \quad y_1 = \frac{28}{7} = 4$$

Substituting $y_1 = 4$ in (2), we get

$4x_1 + 4 - 16 = 0$

$\Rightarrow$ $4x_1 = 12$ $\Rightarrow$ $x_1 = \frac{12}{4} = 3$

$\therefore$ The centre of the required circle is S= $(x_1, y_1) = (3, 4)$

Radius = distance SA = $\sqrt{(3-4)^2 + (4-1)^2} = \sqrt{1+9} = \sqrt{10}$

$\therefore$ The equation of the required circle is, $(x - x_1)^2 + (y - y_1)^2 = r^2$

$\Rightarrow$ $(x-3)^2 + (y-4)^2 = \left(\sqrt{10}\right)^2$

$\Rightarrow$ $x^2 - 6x + 9 + y^2 - 8y + 16 = 10$

$\Rightarrow$ $x^2 + y^2 - 6x - 8y + 15 = 0$

14. Find the equation of the circle whose centre lies on the x - axis and passing through (–2, 3) and (4, 5).

Sol : **First Method**

Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1)

Its centre $(-g, -f)$ lies on the x-axis

$\Rightarrow$ $\boxed{f = 0}$

It passes through (–2, 3)

$\Rightarrow$ $(-2)^2 + 3^2 + 2g(-2) + 2(0)(3) + c = 0$ $[\because f = 0]$

$\Rightarrow$ $13 - 4g + c = 0$ ... (2)

It passes through (4, 5)

$\Rightarrow$ $4^2 + 5^2 + 2g(4) + 0 + c = 0$

$\Rightarrow$ $41 + 8g + c = 0$ ... (3)

Solving (2) & (3) we get

$$\begin{array}{l} 13 - 4g + c = 0 \\ 41 + 8g + c = 0 \\ - \quad - \quad\quad - \\ \hline -28 - 12g = 0 \end{array} \qquad \Rightarrow \qquad -12g = 28$$

$\Rightarrow \quad \boxed{g = \dfrac{28}{-12} = \dfrac{-7}{3}}$

Substituting $g = \dfrac{-7}{3}$ in (2) we get

$$13 - 4\left(\frac{-7}{3}\right) + c = 0 \;\Rightarrow 13 + \frac{28}{3} + c = 0$$

$\Rightarrow \quad \dfrac{39+28}{3} + c = 0$

$\Rightarrow \quad \dfrac{67}{3} + c = 0$

$\Rightarrow \quad c = -\dfrac{67}{3}$

Substituting the values of g, f, c in (1) we get the required circle as

$$x^2 + y^2 + 2\left(\frac{-7}{3}\right)x + 2(0)y - \frac{67}{3} = 0$$

$\Rightarrow \quad 3(x^2 + y^2) - 14x - 67 = 0$

Second Method

Since the centre of the circle lies on x- axis,

let the centre be $S = (x_1, 0)$

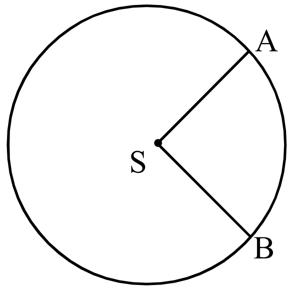
The circle passes through the points A(–2, 3) and B(4, 5)

$\Rightarrow \quad SA = SB$

Squaring on both sides, we get

$SA^2 = SB^2$

$\Rightarrow \quad (x_1 + 2)^2 + (0 - 3)^2 = (x_1 - 4)^2 + (0 - 5)^2$



$\Rightarrow \quad x_1^2 + 4x_1 + 4 + 9 = x_1^2 + 16 - 8x_1 + 25$

$\Rightarrow \quad 4x_1 + 13 + 8x_1 - 41 = 0$

$\Rightarrow \quad 12x_1 - 28 = 0$

$\Rightarrow \quad x_1 = \frac{28}{12} = \frac{7}{3}$

$\therefore$ The centre is $S = (x_1, 0) = \left(\frac{7}{3}, 0\right)$

Radius $= r =$ Distance between S, A $= \sqrt{\left(\frac{7}{3}+2\right)^2 + (0-3)^2} = \sqrt{\left(\frac{7+6}{3}\right)^2 + 9}$

$$= \sqrt{\left(\frac{13}{3}\right)^2 + 9} = \sqrt{\frac{169}{9} + 9} = \sqrt{\frac{250}{9}}$$

$\therefore$ The equation of the required circle is

$$(x - x_1)^2 + (y-0)^2 = \left(\sqrt{\frac{250}{9}}\right)^2$$

$\Rightarrow \quad \left(x - \frac{7}{3}\right)^2 + y^2 = \frac{250}{9}$

$\Rightarrow \quad x^2 + \frac{49}{9} - \frac{14}{3}x + y^2 = \frac{250}{9}$

$\Rightarrow \quad x^2 + y^2 - \frac{14x}{3} = \frac{250}{9} - \frac{49}{9}$

$\Rightarrow \quad \frac{3x^2 + 3y^2 - 14x}{3} = \frac{67}{3}$

$\Rightarrow \quad 3x^2 + 3y^2 - 14x - 67 = 0$

15. If the abscissae of points A, B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and ordinates of A, B are roots of $y^2 + 2py - q^2 = 0$, then find the equation of a circle for which $\overline{AB}$ is a diameter.

Sol : Let $A = (x_1, y_1)$, $B = (x_2, y_2)$

Then x_1 and x_2 are the roots of $x^2 + 2ax - b^2 = 0$ and

y_1 and y_2 are the roots of $y^2 + 2py - q^2 = 0$.

because abscissae of A & B are x_1 and x_2

and ordinates of A & B are y_1 and y_2

Now for the equation $x^2 + 2ax - b^2 = 0$,

Sum of the roots $= x_1 + x_2 = \dfrac{-(2a)}{1} = -2a$

Product of the roots $= x_1 \cdot x_2 = \dfrac{-b^2}{1} = -b^2$

Similarly, for the equation $y^2 + 2py - q^2 = 0$,

sum of the roots $= y_1 + y_2 = \dfrac{-2p}{1} = -2p$

product of the roots $= y_1 \cdot y_2 = \dfrac{-q^2}{1} = -q^2$

Now the equation of the circle with $\overline{AB}$ as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\Rightarrow \quad x^2 - x_1 x - x_2 x + x_1 x_2 + y^2 - y_1 y - y_2 y + y_1 y_2 = 0$

$\Rightarrow \quad x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + (x_1 x_2 + y_1 y_2) = 0$

$\Rightarrow \quad x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

[Note : for the quadratic equation $ax^2 + bx + c = 0$,

Sum of the roots $= \dfrac{-b}{a} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of the roots $= \dfrac{c}{a} = \dfrac{\text{constant}}{\text{coefficient of } x^2}$

16. Find the equation of the circle passing through (0, 0) and making intercepts 4, 3 on x-axis and y-axis respectively.

Sol : Let the required circle be

$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots (1)$

It passes through (0, 0) $\Rightarrow \boxed{c = 0}$

Its x-intercept is 4

$\Rightarrow \quad 2\sqrt{g^2 - c} = 4 \quad \Rightarrow \sqrt{g^2 - 0} = \frac{4}{2}$

$\Rightarrow \quad \sqrt{g^2} = 2 \quad \Rightarrow \pm g = 2$

$\Rightarrow \quad \boxed{g = \pm 2}$

Similarly, its y - intercept is 3

$\Rightarrow \quad 2\sqrt{f^2 - c} = 3 \quad \Rightarrow 2\sqrt{f^2 - 0} = 3 \Rightarrow \sqrt{f^2} = \frac{3}{2}$

$\Rightarrow \quad \pm f = \frac{3}{2} \Rightarrow \boxed{f = \pm\frac{3}{2}}$

substituting the values of g, f, c in (1)

we get the required circle as $x^2 + y^2 \pm 4x \pm 3y = 0$

17. Locate the position of the point P(3, 4) with respect to the circle $S \equiv x^2 + y^2 - 4x - 6y - 12 = 0$

Sol : $S = x^2 + y^2 - 4x - 6y - 12 = 0$, P (3, 4)

$S_{11} = 3^2 + 4^2 - 4(3) - 6(4) - 12 = 9 + 16 - 12 - 24 - 12$

$= -23 < 0$

$\Rightarrow$ P (3, 4) lies inside the circle

18. Find the power of the point P (5, –6) with respect to the circle

$S = x^2 + y^2 + 8x + 12y + 15 = 0$

Sol : $P = (x_1, y_1) = (5, -6)$, $S = x^2 + y^2 + 8x + 12y + 15 = 0$

$\therefore$ power of 'P' with respect to the circle S = 0 is S_{11}

$= 5^2 + (-6)^2 + 8(5) + 12(-6) + 15$

$= 25 + 36 + 40 - 72 + 15$

$= 44$

19. Find the length of tangent from P(1, 3) to the circle

$x^2 + y^2 - 2x + 4y - 11 = 0$

Sol : The length of tangent from $P(1, 3) = (x_1, y_1)$ to the circle $S = x^2 + y^2 - 2x + 4y - 11 = 0$ is $\sqrt{S_{11}}$

$= \sqrt{x_1^2 + y_1^2 - 2x_1 + 4y_1 - 11} = \sqrt{1 + 9 - 2 + 12 - 11} = \sqrt{9} = 3$

20. If the length of the tangent from (2, 5) to the circle

$x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{37}$, then find k.

Sol : Length of the tangent from $P(x_1, y_1) = (2, 5)$ to the circle $S = x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{S_{11}} = \sqrt{37}$

Squaring on both sides we get $S_{11} = 37$

$\Rightarrow \quad x_1^2 + y_1^2 - 5x_1 + 4y_1 + k = 37$

$\Rightarrow \quad 2^2 + 5^2 - 5(2) + 4(5) + k = 37$

$\Rightarrow \quad 4 + 25 - 10 + 20 + k = 37$

$\Rightarrow \quad k = 37 - 39$

$\Rightarrow \quad \boxed{k = -2}$

21. If a point P is moving such that the lengths of tangents drawn from P to the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ are in the ratio 2 : 3, then find the equation of the locus of P.

Sol : Let $P = (x_1, y_1)$

Let $S = x^2 + y^2 - 4x - 6y - 12 = 0$ & $S^1 = x^2 + y^2 + 6x + 18y + 26 = 0$

The length of tangent from P to the circle S = 0 is $\sqrt{S_{11}}$

$$= \sqrt{x_1^2 + y_1^2 - 4x_1 - 6y_1 - 12}$$

The length of tangent from P to the circle $S^1 = 0$ is $\sqrt{S'_{11}}$

$$= \sqrt{x_1^2 + y_1^2 + 6x_1 + 18y_1 + 26}$$

Given that $\sqrt{S_{11}} : \sqrt{S'_{11}} = 2 : 3$

$$\Rightarrow \quad \frac{\sqrt{S_{11}}}{\sqrt{S'_{11}}} = \frac{2}{3}$$

Squaring on both sides we get

$$\frac{S_{11}}{S'_{11}} = \frac{4}{9}$$

$\Rightarrow \quad 9\,S_{11} = 4S'_{11}$

$\Rightarrow \quad 9(x_1^2 + y_1^2 - 4x_1 - 6y_1 - 12) = 4(x_1^2 + y_1^2 + 6x_1 + 18y_1 + 26)$

$\Rightarrow \quad 9x_1^2 + 9y_1^2 - 36x_1 - 54y_1 - 108 - 4x_1^2 - 4y_1^2 - 24x_1 - 72y_1 - 104 = 0$

$\Rightarrow \quad 5x_1^2 + 5y_1^2 - 60x_1 - 126y_1 - 212 = 0$

$\therefore$ The locus of $P(x_1, y_1)$ is

$5x^2 + 5y^2 - 60x - 126y - 212 = 0$

22. Find the equation of the tangent to $x^2 + y^2 - 2x + 4y = 0$ at $(3, -1)$. Also find the equation of tangent parallel to it.

Sol : The given circle is $S = x^2 + y^2 - 2x + 4y = 0$ ... (1)

Let $P = (x_1, y_1) = (3, -1)$

$S_{11} = 3^2 + (-1)^2 - 2(3) + 4(-1) = 9 + 1 - 6 - 4 = 10 - 10 = 0$

$\Rightarrow$ P lies on the circle (1)

$\therefore$ Equation of tangent at P is $S_1 = 0$

$\Rightarrow \quad x\,x_1 + y\,y_1 + g\,(x + x_1) + f\,(y + y_1) = 0$

$\Rightarrow \quad x(3) + y(-1) + (-1)\,(x + 3) + 2\,(y - 1) = 0$

$\Rightarrow \quad 3x - y - x - 3 + 2y - 2 = 0$

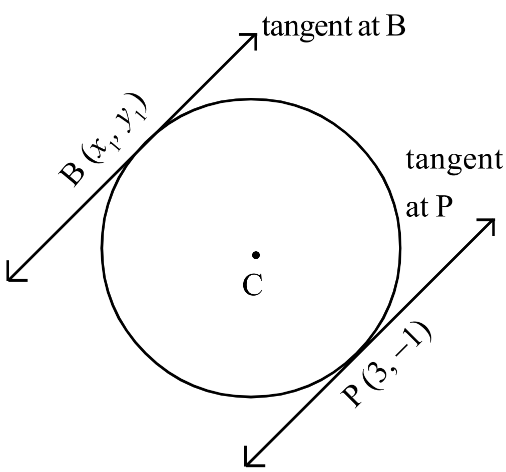
$\Rightarrow \quad 2x + y - 5 = 0$ ... (2)

The centre of the circle $C = (-g, -f)$

$\Rightarrow \quad C = (1, -2)$

Let B be the other end of the diameter $\overline{PCB}$

Let $B = (x_1, y_1)$ then C = mid point of P, B



$$\Rightarrow \quad (1, -2) = \left(\frac{3 + x_1}{2}, \frac{-1 + y_1}{2}\right)$$

$$\Rightarrow \quad 1 = \frac{3 + x_1}{2}, \qquad -2 = \frac{-1 + y_1}{2}$$

$\Rightarrow \quad 2 = 3 + x_1 \qquad \Rightarrow -4 = -1 + y_1$

$\Rightarrow \quad \boxed{x_1 = -1} \qquad \Rightarrow \boxed{y_1 = -3}$

$\therefore$ The other end of the diameter is B = (x_1, y_1)

$$= (-1, -3)$$

$\therefore$ The tangent parallel to (2), will pass through B.

Any line parallel to (2) is $2x + y + k = 0$ ... (3)

It passes through B

$\Rightarrow$ $2(-1) + (-3) + k = 0$

$\Rightarrow$ $\boxed{k = 5}$

Substituting $k = 5$ in (3), we get the required tangent parallel to (2) as $2x + y + 5 = 0$.

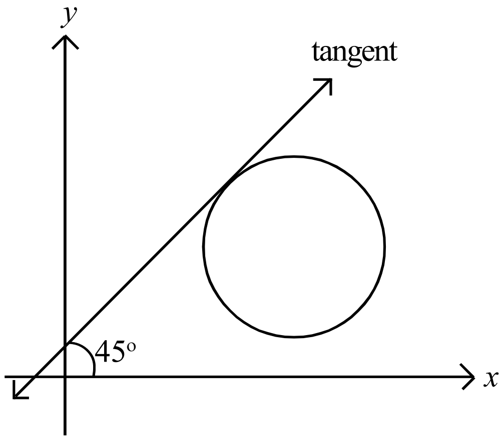
23. Find the equation of the tangents to the circle $x^2 + y^2 - 4x - 6y + 3 = 0$ which makes an angle 45° with x-axis.

Sol : Given circle is $S = x^2 + y^2 - 4x - 6y + 3 = 0$. Comparing with the standard equation,

$x^2 + y^2 + 2gx + 2fy + c = 0$, we get $2g = -4$, $2f = -6$, $c = 3$

$\Rightarrow$ $g = -2, f = -3, c = 3$

radius $= r = \sqrt{g^2 + f^2 - c} = \sqrt{4+9-3} = \sqrt{10}$



Given that the tangent makes an angle 45° with x-axis

$\Rightarrow$ Slope of tangent $= m = \text{Tan } 45^\circ = 1$

$\therefore$ The equation of required tangent is

$$y + f = m(x + g) \pm r\sqrt{1+m^2}$$

$\Rightarrow \quad y-3=1(x-2) \pm \sqrt{10}\ \sqrt{1+1}$

$\Rightarrow \quad x-y+1 \pm 2\sqrt{5}=0$

24. Find the length of the chord intercepted by the circle

$x^2 + y^2 + 8x - 4y - 16 = 0$ on the line $3x - y + 4 = 0$

Sol : Given circle is

$S = x^2 + y^2 + 8x - 4y - 16 = 0$

Comparing with the standard equation $x^2 + y^2 + 2gx + 2fy + c = 0$,

we get $2g = 8$, $2f = -4$, $c = -16$

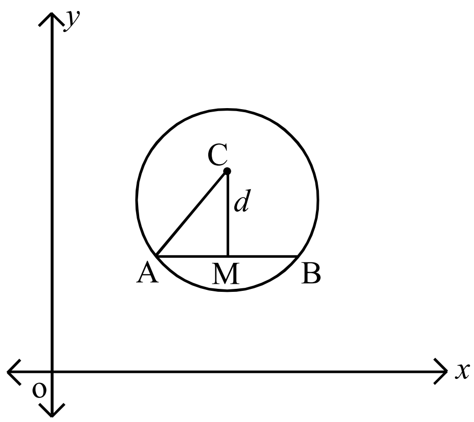
$\Rightarrow \quad g = 4, f = -2, c = -16$

radius $= r = \sqrt{g^2 + f^2 - c}$

$= \sqrt{4^2 + (-2)^2 + 16}$

$= \sqrt{36} = 6$

centre $= C = (-g, -f) = (-4, 2) = (x_1, y_1)$



let the equation of chord $\overline{AB}$ be $3x - y + 4 = 0$ ... (1)

$CM = d =$ perpendicular distance from the centre C to the chord (1)

$= \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ (formula)

$\therefore \quad ax + by + c = 3x - y + 4 = 0$, $a = 3$, $b = -1$, $c = 4$

$$= \frac{|-4(3)+(-1)(2)+4|}{\sqrt{3^2+(-1)^2}}$$

$$= \frac{|-10|}{\sqrt{10}} = \frac{\sqrt{10}\sqrt{10}}{\sqrt{10}} = \sqrt{10}$$

$\therefore$ length of the chord AB

$= 2\sqrt{r^2 - d^2}$

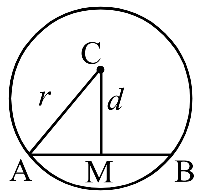
$= 2\sqrt{6^2 - \left(\sqrt{10}\right)^2}$

$= 2\sqrt{36-10}$

$= 2\sqrt{26}$ units

25. Find the length of the chord formed by $x^2 + y^2 = a^2$ on the line $x\cos\alpha + y\sin\alpha = p$

Sol. The given circle is $x^2 + y^2 = a^2$



Its centre is $(0, 0) = C$

and radius $= r = a$

Given equation of chord is

$$x\cos\alpha + y\sin\alpha - p = 0 \quad \text{... (1)}$$

comparing with $ax + by + c = 0$, we get

$a = \cos\alpha$, $b = \sin\alpha$, $c = -p$

$\therefore$ $d = CM$ = length of the perpendicular from the centre $C = (0, 0) = (x_1, y_1)$ to the chord (1)

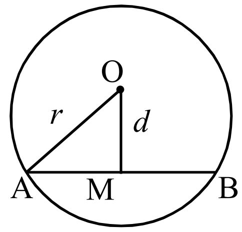
$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} \text{ (formula)}$$

$$= \left|\frac{\cos\alpha\,(0) + \sin\alpha\,(0) - p}{\sqrt{\cos^2\alpha + \sin^2\alpha}}\right| = |-p| = p$$

$\therefore$ Length of the chord AB $= 2\sqrt{r^2 - d^2}$

$= 2\sqrt{a^2 - p^2}$ units

26. If $x^2 + y^2 = c^2$ and $\frac{x}{a} + \frac{y}{b} = 1$ intersect at A and B then find $\overline{AB}$. Hence deduce the condition that the line touches the circle.



Sol. The given circle is $x^2 + y^2 = c^2$

Its centre is O = (0, 0)

radius = $r = c$

The equation of chord AB is $\frac{x}{a} + \frac{y}{b} = 1$ ... (1)

d = perpendicular distance from the centre O (0, 0) to the chord (1)

= OM

$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ (formula) where $(x_1, y_1) = (0, 0)$

$= \frac{\left|\frac{0}{a} + \frac{0}{b} - 1\right|}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}}$ [$\because$ The line (1) is $\frac{1}{a} \cdot x + \frac{1}{b} \cdot y - 1 = 0$]

$= \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$

Now the length of the chord $\overline{AB}$

$= 2\sqrt{r^2 - d^2}$

$= 2\sqrt{c^2 - \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}}$

$= 2 \times \sqrt{c^2 - \frac{a^2 b^2}{b^2 + a^2}}$

The line (1) will be a tangent or touches the circle, if this length of chord is zero.

$\Rightarrow \quad 2\sqrt{c^2 - \dfrac{a^2 b^2}{b^2 + a^2}} = 0$

$\Rightarrow \quad c^2 - \dfrac{a^2 b^2}{b^2 + a^2} = 0$

$\Rightarrow \quad c^2 = \dfrac{a^2 b^2}{b^2 + a^2}$

$\Rightarrow \quad \dfrac{1}{c^2} = \dfrac{a^2 + b^2}{a^2 b^2}$

$\Rightarrow \quad \dfrac{1}{c^2} = \dfrac{a^2}{a^2 b^2} + \dfrac{b^2}{a^2 b^2}$

$\Rightarrow \quad \dfrac{1}{c^2} = \dfrac{1}{b^2} + \dfrac{1}{a^2}$ is the required condition

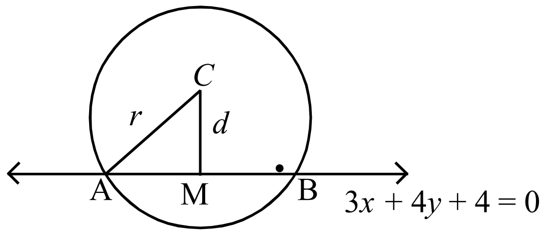
27. Find the equation of the circle with centre (–2, 3) and cutting a chord of length 2 units on $3x + 4y + 4 = 0$

Sol : Let the required cricle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1)

Its centre is $(-g, -f) = (-2, 3)$ (given)

$\Rightarrow \quad -g = -2, -f = 3$

$\Rightarrow \quad \boxed{g = 2}, \boxed{f = -3}$



equation of the chord $\overline{AB}$ is

$3x + 4y + 4 = 0$ ... (2)

$\therefore \quad$ d = perpendicular distance from the centre (–2, 3) to the chord (2)

$= \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ (formula) $\quad (x_1, y_1) = (-2, 3)$

$= \dfrac{|3(-2) + 4(3) + 4|}{\sqrt{3^2 + 4^2}}$

$= \dfrac{10}{5} = 2$

$\therefore$ The length of the chord AB is 2. (Given)

$\Rightarrow$ $2\sqrt{r^2 - d^2} = 2$

$\Rightarrow$ $\sqrt{r^2 - d^2} = 1$

$\Rightarrow$ $r^2 - d^2 = 1$

$\Rightarrow$ $r^2 = 1 + d^2$

$\Rightarrow$ $g^2 + f^2 - c = 1 + 2^2$ $[\because\ d = 2]$

$\Rightarrow$ $(2)^2 + (-3)^2 - c = 5$

$\Rightarrow$ $\boxed{c = 8}$

Substituting the values of g, f, c in (1), we get the required circle as

$x^2 + y^2 + 4x - 6y + 8 = 0$

28. Find the equation of the circle with centre (2, 3) and touching the line $3x - 4y + 1 = 0$

Sol : The centre of the circle is $(a, b) = (2, 3)$

Since it touches the line $3x - 4y + 1 = 0$,

The line $3x - 4y + 1 = 0$ ... (1)

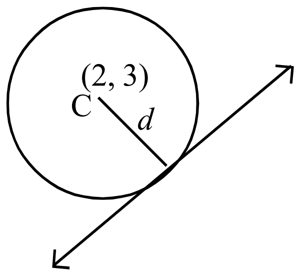
is a tangent to the circle

$\Rightarrow$ radius $= d =$ perpendicular distance from the centre (2, 3) to the tangent (1)

$= \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ (formula) where $(x_1, y_1) = (2, 3)$

$= \dfrac{|3(2) - 4(3) + 1|}{\sqrt{3^2 + (-4)^2}}$

$= \dfrac{|-5|}{5} = 1$



$\therefore$ The equation of the required circle is

$(x - a)^2 + (y - b)^2 = r^2$

$\Rightarrow$ $(x - 2)^2 + (y - 3)^2 = 1^2$

$\Rightarrow$ $x^2 + y^2 - 4x - 6y + 12 = 0$

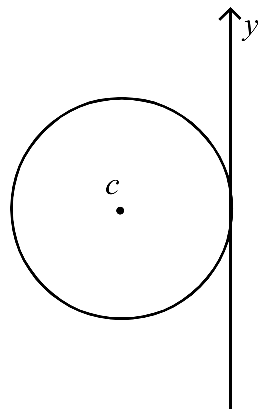
29. Find the equation of the circle with centre (–3, 4) and touching y - axis

Sol : Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

Its centre = $(-g, -f) = (-3, 4)$ given

$\Rightarrow$ $-g = -3, -f = 4$

$\Rightarrow$ $\boxed{g = 3}$, $\boxed{f = -4}$



Since the circle touches the y - axis, $f^2 = c$ (condition)

$\Rightarrow$ $c = f^2 = (-4)^2 = 16$

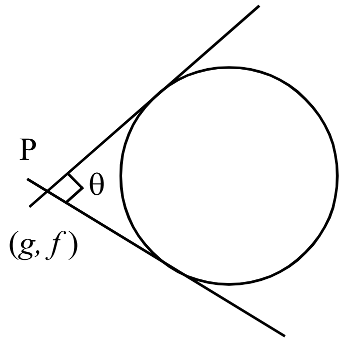
$\Rightarrow$ $\boxed{c = 16}$

$\therefore$ The required circle is $x^2 + y^2 + 2(3)x + 2(-4)y + 16 = 0$

$\Rightarrow$ $x^2 + y^2 + 6x - 8y + 16 = 0$

30. Find the condition that the tangents drawn from the external point (g, f) to the circle S = 0 are perpendicular to each other.

Sol : We know that if θ is the angle between the tangents drawn from an external point $P(x_1, y_1)$ to the circle S = 0, then $\tan\left(\frac{\theta}{2}\right) = \frac{r}{\sqrt{S_{11}}}$



If they are perpendicular, then $\theta = 90^\circ$

Given $P(x_1, y_1) = (g, f)$

$\text{Tan}\left(\frac{90}{2}\right) = \frac{r}{\sqrt{S_{11}}}$

$\Rightarrow$ $\text{Tan } 45^\circ = \frac{r}{\sqrt{S_{11}}}$ $\Rightarrow 1 = \frac{r}{\sqrt{S_{11}}}$

$\Rightarrow$ $\sqrt{S_{11}} = r$

Squaring on both sides, we get

$S_{11} = r^2$

$\Rightarrow$ $x_1^2 + y_1^2 + 2g\,x_1 + 2fy_1 + c = g^2 + f^2 - c$

$\Rightarrow$ $g^2 + f^2 + 2g\,(g) + 2f\,(f) + c = g^2 + f^2 - c$

$\Rightarrow$ $2g^2 + 2f^2 + 2c = 0$

$\Rightarrow \quad 2(g^2 + f^2 + c) = 0$

$\Rightarrow \quad g^2 + f^2 + c = 0$ is the required condition.

31. Find the chord of contact of (2, 5) with respect to the circle $x^2 + y^2 - 5x + 4y - 2 = 0$

Sol : Let $P = (x_1, y_1) = (2, 5)$

The circle is $S = x^2 + y^2 - 5x + 4y - 2 = 0$

The chord of contact of P, w.r.t the circle $S = 0$ is $S_1 = 0$

$\Rightarrow \quad x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow \quad x(2) + y(5) - \frac{5}{2}(x+2) + 2(y+5) - 2 = 0$

$\Rightarrow \quad 2x + 5y - \frac{5x}{2} - 5 + 2y + 10 - 2 = 0$

$\Rightarrow \quad 2x + 7y - \frac{5x}{2} + 3 = 0 \quad \Rightarrow \quad x - 14y - 6 = 0$

32. Find the equation of the polar of (2, 3) with respect to the circle $x^2 + y^2 + 6x + 8y - 96 = 0$

Sol : Let $P = (x_1, y_1) = (2, 3)$

The circle is $S = x^2 + y^2 + 6x + 8y - 96 = 0$

The polar of $P = (x_1, y_1)$ with respect to the circle $S = 0$ is $S_1 = 0$

$\Rightarrow$ The polar of P(2, 3) is

$x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c = 0$

$\Rightarrow \quad x(2) + y(3) + 3(x + 2) + 4(y + 3) - 96 = 0$

$\Rightarrow \quad 2x + 3y + 3x + 6 + 4y + 12 - 96 = 0$

$\Rightarrow \quad 5x + 7y - 78 = 0$

33. Show that (4, 2) and (3, –5) are conjugate points with respect to the circle $x^2 + y^2 - 3x - 5y + 1 = 0$

Sol : Let $P = (x_1, y_1) = (4, 2)$, $Q = (x_2, y_2) = (3, -5)$

The circle is $S = x^2 + y^2 - 3x - 5y + 1 = 0$

Now $\quad S_{12} = x_1 x_2 + y_1 y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c$

$$= 4(3) + 2(-5) - \frac{3}{2}(4+3) - \frac{5}{2}(2-5) + 1$$

$$= 12 - 10 - \frac{21}{2} + \frac{15}{2} + 1$$

$$= 3 - \frac{21}{2} + \frac{15}{2} = \frac{6 - 21 + 15}{2} = \frac{0}{2} = 0$$

Since $S_{12} = 0$, the points P and Q are conjugate points.

34. Find the pole of $x + y + 2 = 0$ with respect to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$

Sol. To find the pole of the line $x + y + 2 = 0$ ... (1)

with respect to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ ... (2)

comparing (1) with $lx + my + n = 0$

we get $l = 1, m = 1, n = 2$

because, (1) is $1.x + 1.y + 2 = 0$

comparing (2) with the standard equation $x^2 + y^2 + 2gx + 2fy + c = 0$

we get $2g = -4$, $2f = 6$, $c = -12$

$\Rightarrow$ $g = -2$, $f = 3$, $c = -12$

radius $= r = \sqrt{g^2 + f^2 - c} = \sqrt{4+9+12} = 5$

$\therefore$ The pole of $lx + my + n = 0$ w.r.t the circle $S = 0$

is $\left(-g + \dfrac{lr^2}{lg + mf - n}, -f + \dfrac{mr^2}{lg + mf - n}\right)$

$\therefore$ The pole of (1) is $= \left(2 + \dfrac{1\times 25}{1(-2)+1(3)-2}, -3 + \dfrac{1\times 25}{1(-2)+1(3)-2}\right)$

$$= \left(2 + \frac{25}{-1}, -3 + \frac{25}{-1}\right)$$

$$= (2 - 25, -3 - 25)$$

$$= (-23, -28)$$

35. If $(4, k)$ and $(2, 3)$ are conjugate points with respect to the circle $x^2 + y^2 = 17$, then find k.

Sol : Let $P = (x_1, y_1) = (4, k)$, $Q = (x_2, y_2) = (2, 3)$

given circle is $S = x^2 + y^2 - 17 = 0$

It is given that P and Q are conjugate points

$\Rightarrow$ $S_{12} = 0$

$\Rightarrow$ $x_1 x_2 + y_1 y_2 - 17 = 0$

$\Rightarrow$ $4(2) + k(3) - 17 = 0$

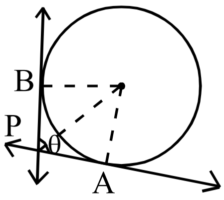
$\Rightarrow$ $8 + 3k - 17 = 0$

$\Rightarrow$ $3k = 9$

$\Rightarrow$ $\boxed{k = 3}$

36. Find the angle between the tangents drawn from (3, 2) to the circle

$$x^2 + y^2 - 6x + 4y - 2 = 0$$



Sol.: Let $P = (x_1, y_1) = (3, 2)$ & Circle $S = x^2 + y^2 - 6x + 4y - 2 = 0$. We know that if '$\theta$' is the angle between the tangents drawn from $P(x_1, y_1)$ to the circle $S = 0$, then

$$\tan\left(\frac{\theta}{2}\right) = \frac{r}{\sqrt{S_{11}}}$$

$$\Rightarrow \quad \tan\left(\frac{\theta}{2}\right) = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{S_{11}}}$$

$$= \frac{\sqrt{9+4+2}}{\sqrt{3^2 + 2^2 - 6(3) + 4(2) - 2}}$$

because $S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$$\therefore \quad \operatorname{Tan}\left(\frac{\theta}{2}\right) = \frac{\sqrt{15}}{\sqrt{1}} = \sqrt{15}$$

$$\Rightarrow \quad \frac{\theta}{2} = \operatorname{Tan}^{-1}(\sqrt{15})$$

$\Rightarrow \quad \theta = 2\operatorname{Tan}^{-1}(\sqrt{15})$, is the angle between the tangents.

(or) $\quad \operatorname{Tan}\dfrac{\theta}{2} = \sqrt{15}$

$$\Rightarrow \cos\theta = \left|\frac{1 - \operatorname{Tan}^2 \theta/2}{1 + \operatorname{Tan}^2 \theta/2}\right| = \left|\frac{1-15}{1+15}\right| = \left|\frac{-14}{16}\right| = \frac{+7}{8}$$

$$\Rightarrow \theta = \operatorname{Cos}^{-1}\left(\frac{7}{8}\right) \text{ Ans.}$$

37. Find the internal centre of similitude for the circles

$x^2 + y^2 + 6x - 2y + 1 = 0$ and $x^2 + y^2 - 2x - 6y + 9 = 0$

Sol : Let the given circles be

$$S = x^2 + y^2 + 6x - 2y + 1 = 0$$

and $S^1 = x^2 + y^2 - 2x - 6y + 9 = 0$

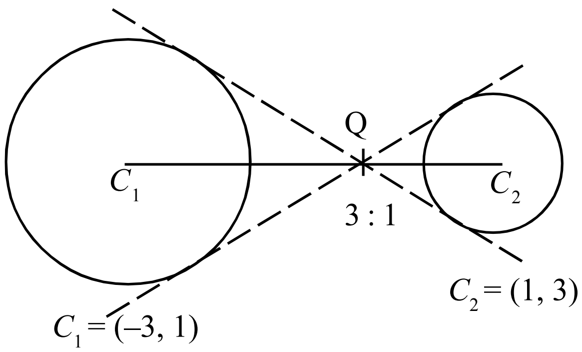
For the circle $S = 0$,	For the circle $S^1 = 0$
centre $= C_1 = (-3, 1)$	centre $= C_2 = (1, 3)$
radius $= r_1 = \sqrt{9+1-1}$	radius $= r_2 = \sqrt{1+9-9}$
$= \sqrt{9} = 3$	$= 1.$

Distance $\overline{C_1C_2} = \sqrt{(1+3)^2 + (3-1)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

$r_1 + r_2 = 4$

$\Rightarrow$ $C_1C_2 > r_1 + r_2$

$\Rightarrow$ The two circles are non - intersecting circles.



The internal centre of similitude divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2 = 3 : 1$ internally.

$\Rightarrow$ The internal centre of similitude = Q

$$= \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

$$= \left(\frac{3(1)+1(-3)}{3+1}, \frac{3(3)+1(1)}{3+1}\right)$$

$$= \left(0, \frac{10}{4}\right) = \left(0, \frac{5}{2}\right)$$

38. Find the external centre of similitude for the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 = 4$

Sol : Let the given circles be

$$S = x^2 + y^2 - 2x - 6y + 9 = 0$$

and $S^1 = x^2 + y^2 - 4 = 0$

for the circle S = 0,

centre = $C_1 = (1, 3)$

radius = $r_1 = \sqrt{1+9-9} = 1$

for the circle $S^1 = 0$

centre = $C_2 = (0, 0)$

radius = $r_2 = \sqrt{4} = 2$

Distance $\overline{C_1C_2} = \sqrt{(1-0)^2+(3-0)^2} = \sqrt{10}$

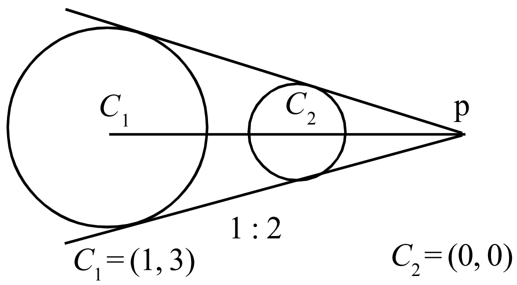
$r_1 + r_2 = 1 + 2 = 3.$

$\overline{C_1C_2} > r_1 + r_2$

$\therefore$ The circles are non - intersecting.

The external centre of similitude, P is the point of intersection of direct common tangents and divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2 = 1 : 2$ externally.

$\therefore$ The external centre of similitude



$$P = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

$$= \left(\frac{1(0)-2(1)}{1-2}, \frac{1(0)-2(3)}{1-2}\right)$$

$$= \left(\frac{-2}{-1}, \frac{-6}{-1}\right)$$

$$= (2, 6)$$

39. Show that the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ touch each other. Also find the point of contact and the equation of common tangent at this point of contact.

Sol : Let the given circles be

$$S = x^2 + y^2 - 4x - 6y - 12 = 0$$

and $\quad S^1 = x^2 + y^2 + 6x + 18y + 26 = 0$

for the circle S = 0,

centre = $C_1 = (2, 3)$

radius = $r_1 = \sqrt{4+9+12}$

$= \sqrt{25}$

$= 5$

for the circle $S^1 = 0$

centre = $C_2 = (-3, -9)$

radois = $r_2 = \sqrt{9+81-26}$

$= \sqrt{64}$

$= 8$

Distance $\overline{C_1C_2} = \sqrt{(-3-2)^2+(-9-3)^2}$

$= \sqrt{25+144} = \sqrt{169} = 13$

$r_1 + r_2 = 5 + 8 = 13 = \overline{C_1C_2}$

$\therefore$ $\overline{C_1C_2} = r_1 + r_2$

$\Rightarrow$ The two circles touch each other externally

The common tangent is the radical axis, $S - S' = 0$

$\Rightarrow$ $x^2+y^2-4x-6y-12-x^2-y^2-6x-18y-26=0$

$\Rightarrow$ $-10x - 24y - 38 = 0$

$\Rightarrow$ $-2(5x + 12y + 19) = 0$

$\Rightarrow$ $5x + 12y + 19 = 0$ is the equation of common tangent at the point of contact.

To find the point of contact of two circles : -

Let P (h, k) be the point of contact of the circles.

Then P is the foot of the perpendicular drawn from $C_1 = (2, 3) = (x_1, y_1)$

to the tangent $5x + 12y + 19 = 0$

$$\Rightarrow \quad \frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{-(ax_1+by_1+c)}{a^2+b^2}$$

$$\Rightarrow \quad \frac{h-2}{5} = \frac{k-3}{12} = \frac{-(5(2)+12(3)+19)}{5^2+12^2}$$

$$\Rightarrow \quad \frac{h-2}{5} = \frac{k-3}{12} = \frac{-65}{169}$$

$$\Rightarrow \quad \frac{h-2}{5} = \frac{k-3}{12} = \frac{-5}{13}$$

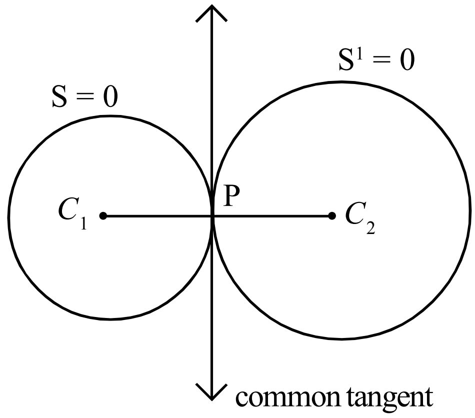
$$\Rightarrow \quad \frac{h-2}{5} = \frac{-5}{13}, \frac{k-3}{12} = \frac{-5}{13}$$

$$\Rightarrow \quad h-2 = \frac{-25}{13}, \quad k-3 = \frac{-60}{13}$$

$h = 2 - \frac{25}{13}$ | $k = 3 - \frac{60}{13}$

$= \frac{26-25}{13}$ | $= \frac{39-60}{13}$

$= \frac{1}{13}$ | $= -\frac{21}{13}$



$\therefore$ The point of contact of the two circles is

$$= (h,k) = \left(\frac{1}{13}, \frac{-21}{13}\right)$$

Second method to find the point of contact of the two circles

Since the circle touch each other externally, their point of contact is the internal centre of similitude P which divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2 = 5 : 8$ internally

$\therefore$ The point of contact of the circles

$$= \left(\frac{5(-3)+8(2)}{5+8}, \frac{5(-9)+8(3)}{5+8}\right)$$

$$= \left(\frac{-15+16}{13}, \frac{-45+24}{13}\right)$$

$$= \left(\frac{1}{13}, \frac{-21}{13}\right)$$

40. Show that the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $5(x^2 + y^2) - 8x - 14y - 32 = 0$, touch each other and find their point of contact.

Sol: Let the circles be

$$S = x^2 + y^2 - 4x - 6y - 12 = 0$$

and $S^1 = x^2 + y^2 - \frac{8}{5}x - \frac{14}{5}y - \frac{32}{5} = 0$ (standard form) for the circle S = 0,

$2g = -4,\ 2f = -6,\ c = -12$

$\Rightarrow\ g = -2,\ f = -3,\ c = -12$

$\therefore$ centre $= C_1 = (-g, -f) = (2, 3)$

radius $= r_1 = \sqrt{g^2 + f^2 - c}$

$= \sqrt{4+9+12}$

$= 5$

for the circle $S^1 = 0$

$2g^1 = \frac{-8}{5},\ 2f^1 = \frac{-14}{5},\ c^1 = \frac{-32}{5}$

$\Rightarrow\ g^1 = \frac{-4}{5},\ f^1 = \frac{-7}{5},\ c^1 = \frac{-32}{5}$

5 : 8

centre $= \left(\frac{4}{5}, \frac{7}{5}\right) = C_2$

$C_1 = (2.\ 3)$ $\quad C_2 = (-3, -9)$

radois $= r_2 = \sqrt{\left(g^1\right)^2 + \left(f^1\right)^2 - c^1}$

$= \sqrt{\frac{16}{25} + \frac{49}{25} + \frac{32}{5}}$

$= \sqrt{\frac{16+49+160}{25}}$

$= \sqrt{\frac{225}{25}} = \sqrt{9} = 3$

Distance $C_1C_2 = \sqrt{\left(\frac{4}{5} - 2\right)^2 + \left(\frac{7}{5} - 3\right)^2}$

$= \sqrt{\left(\frac{-6}{5}\right)^2 + \left(\frac{-8}{5}\right)^2}$

$= \sqrt{\frac{36}{25} + \frac{64}{25}}$

$= \sqrt{\frac{36+64}{25}}$

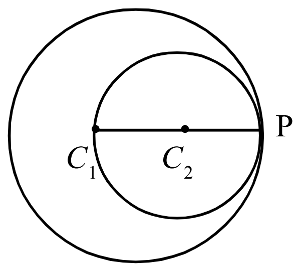
$= \sqrt{\frac{100}{25}} = \sqrt{4} = 2$

$r_1 + r_2 = 5 + 3 = 8$

$r_1 - r_2 = 5 - 3 = 2 = C_1C_2$

Since $C_1C_2 = |r_1 - r_2|$, the two circles touch each other internally

The point of contact of the two circles is the external centre of similitude, P, which divides $\overline{C_1C_2}$ externally in the ratio $r_1 : r_2 = 5 : 3$



$\therefore \quad P = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$

$$= \left(\frac{5\left(\frac{4}{5}\right) - 3(2)}{5-3}, \frac{5\left(\frac{7}{5}\right) - 3(3)}{5-3}\right)$$

$$= \left(\frac{4-6}{2}, \frac{7-9}{2}\right) = \left(\frac{-2}{2}, \frac{-2}{2}\right)$$

$= (-1, -1)$

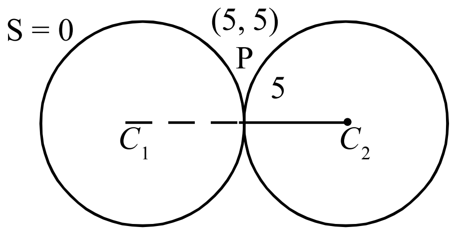
$\therefore$ The point of contact of the two circles $= (-1, -1)$

41. Find the equation of the circle which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at (5, 5) with radius 5

Sol : Let the given circle be $S = x^2 + y^2 - 2x - 4y - 20 = 0$

Its centre is $C_1 = (-g, -f)$

$= (1, 2)$



$2g = -2 \quad \Rightarrow \quad g = -1$

$2f = -4 \quad \Rightarrow f = -2$

$c = -20$

radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{1 + 4 + 20}$

$= \sqrt{25} = 5$

Radius of S = 0 is 5 and the radius of required circle is also 5.

$\Rightarrow$ The two circles touch at P = (5, 5) externally.

So, let the centre of the required circle be $(x_1, y_1) = C_2$

Then P is the internal centre of similitude which divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2 = 5 : 5 = 1 : 1$ internally.

$\Rightarrow$ P is the mid point of $C_1 C_2$

$\Rightarrow$ $(5,5) = \left(\dfrac{1+x_1}{2}, \dfrac{2+y_1}{2}\right)$

$\Rightarrow$ $\dfrac{1+x_1}{2} = 5, \dfrac{2+y_1}{2} = 5$

$\Rightarrow$ $x_1 = 10 - 1, \quad y_1 = 10 - 2$

$x_1 = 9, \quad y_1 = 8$

$\therefore$ The centre of the required circle is $(x_1, y_1) = (9, 8)$ and radius $r = 5$

The equation of required circle is

$$(x-x_1)^2 + (y-y_1)^2 = r^2$$

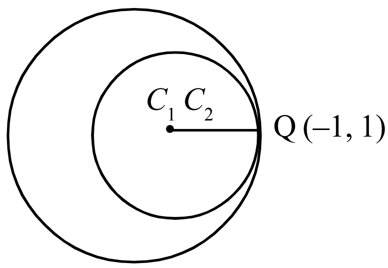
$$(x-9)^2 + (y-8)^2 = 5^2$$

$\Rightarrow$ $x^2 - 18x + 81 + y^2 - 16y + 64 - 25 = 0$

$$x^2 + y^2 - 18x - 16y + 120 = 0$$

42. Find the equation of the circle which touches $x^2 + y^2 - 4x + 6y - 12 = 0$ at $(-1, 1)$ internally with a radius of 2.

Sol : Given circle is



$S = x^2 + y^2 - 4x + 6y - 12 = 0$

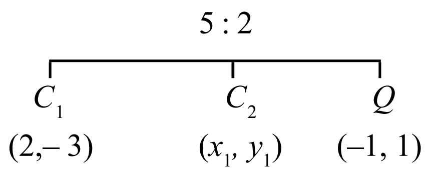
$2g = -4 \qquad \Rightarrow g = -2$

$2f = 6 \qquad \Rightarrow f = 3$

$c = -12$

Its centre $C_1 = (-g, -f) = (2, -3)$

radius $r_1 = \sqrt{4+9+12} = \sqrt{25} = 5$



Let the centre of the required circle be $C_2 = (x_1, y_1)$ whose radius is 2, and touches the circle $S = 0$ internally.

Let $Q = (-1, 1)$ be the point of contact of the two circles.

Then Q is the external centre of similitude which divides C_1, C_2 externally

in the ratio $r_1 : r_2 = 5 : 2$

$\therefore$ $Q = \left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n}\right)$

$\Rightarrow \quad (-1, 1) = \left(\dfrac{5x_1 - 2(2)}{5-2}, \dfrac{5y_1 - 2(-3)}{5-2}\right)$

$\Rightarrow \quad (-1, 1) = \left(\dfrac{5x_1 - 4}{3}, \dfrac{5y_1 + 6}{3}\right)$

$\Rightarrow \quad \dfrac{5x_1 - 4}{3} = -1, \dfrac{5y_1 + 6}{3} = 1$

$\Rightarrow \quad 5x_1 - 4 = -3$ | $5y_1 + 6 = 3$

$\Rightarrow \quad 5x_1 = -3 + 4$ | $5y_1 = 3 - 6$

$\Rightarrow \quad x_1 = \dfrac{1}{5}$ | $5y_1 = -3$

$y_1 = \dfrac{-3}{5}$

$\therefore$ The centre of the required circle is $= (x_1, y_1) = \left(\dfrac{1}{5}, \dfrac{-3}{5}\right)$

$\therefore$ The equation of the required circle with radius 2, is

$(x - x_1)^2 + (y - y_1)^2 == 2^2$

$\Rightarrow \quad \left(x - \dfrac{1}{5}\right)^2 + \left(y + \dfrac{3}{5}\right)^2 = 4$

$\Rightarrow \quad x^2 + \dfrac{1}{25} - \dfrac{2}{5}x + y^2 + \dfrac{9}{25} + \dfrac{6}{5}y - 4 = 0$

$\Rightarrow \quad x^2 + y^2 - \dfrac{2}{5}x + \dfrac{6}{5}y + \dfrac{1}{25} + \dfrac{9}{25} - 4 = 0$

$\Rightarrow \quad x^2 + y^2 - \dfrac{2}{5}x + \dfrac{6}{5}y - \dfrac{18}{5} = 0$

$\Rightarrow \quad 5x^2 + 5y^2 - 2x + 6y - 18 = 0$ is the required circle.

43. Find the pair of tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and hence deduce a condition for these tangents to be perpendicular.

Sol : Let the given circle be

$S = x^2 + y^2 + 2gx + 2fy + c = 0$

Let P $(0, 0) = (x_1, y_1)$, $S_{11} = 0^2 + 0^2 + 2g(0) + 2f(0) + c = c$

The equation of pair of tangents drawn from P to the circle S = 0 is $S_1^2 = SS_{11}$

$\Rightarrow \quad [x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c]^2 = (x^2 + y^2 + 2gx + 2fy + c)(c)$

$\Rightarrow \quad [x(0) + y(0) + gx + fy + c]^2 = (c)(x^2 + y^2 + 2gx + 2fy + c)$

$\Rightarrow \quad (gx + fy + c)^2 = cx^2 + cy^2 + 2gcx + 2fcy + c^2$

$\Rightarrow \quad g^2x^2 + f^2y^2 + c^2 + 2gfxy + 2fcy + 2gcx - cx^2 - cy^2 - 2gcx - 2fcy - c^2 = 0$

$\Rightarrow \quad (g^2 - c)x^2 + (f^2 - c)y^2 + 2gfxy = 0$

or $(gx + fy)^2 = c(x^2 + y^2)$

Now this pair of tangents (pair of straight lines) are perpendiucular,
if coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow \quad (g^2 - c) + (f^2 - c) = 0$

$\Rightarrow \quad g^2 + f^2 = 2c$ is the condition for the pair of tangents to be perpendicular

44. Find the direct common tangents of the circles $x^2 + y^2 + 22x - 4y - 100 = 0$ and $x^2 + y^2 - 22x + 4y + 100 = 0$

Sol.: Let $S = x^2 + y^2 + 22x - 4y - 100 = 0$ and

$S' = x^2 + y^2 - 22x + 4y + 100 = 0$ be the given circles.

For the circle $S = 0$

$2g = 22,\ 2f = -4,\ c = -100$

$\therefore$ centre $C_1 = (-g, -f) = (-11, 2)$

$r_1 = \text{radius} = \sqrt{g^2 + f^2 - c}$

$= \sqrt{121 + 4 + 100}$

$= \sqrt{225}$

$= 15$

For the circle $S' = 0$,

$2g^1 = -22,\ 2f' = 4,\ c' = 100$

centre $= C_2 = (-g', -f') = (+11, -2)$

$r_2 = \text{radius} = \sqrt{(-11)^2 + 2^2 - 100}$

$= \sqrt{121 + 4 - 100}$

$= \sqrt{25}$

$= 5$

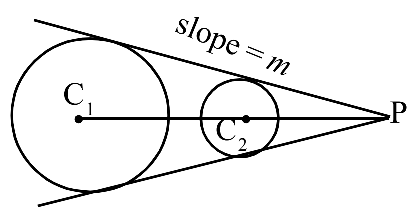
Distance $C_1C_2 = \sqrt{(11+11)^2 + (-2-2)^2}$

$= \sqrt{484 + 16}$

$= \sqrt{500}$

$= \sqrt{5 \times 100} = 10\sqrt{5}$

$\simeq 10(2.2) = 22$



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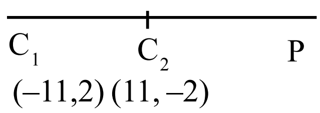


$r_1 + r_2 = 15 + 5 = 20 < C_1C_2$

$\therefore \ C_1C_2 > r_1 + r_2.$

$\Rightarrow$ The two circle are non - intersecting circles.

The direct common tangents are drawn

from the external centre of similitude P, which divides $\overline{C_1C_2}$ in the ratio :

$r_1 : r_2 = 15 : 5 = 3 : 1$ externally.

$$\therefore\ P = \left(\frac{mx_2 - nx_1}{m-n},\ \frac{my_2 - ny_1}{m-n}\right)$$

$$= \left(\frac{3(11)-1(-11)}{3-1},\ \frac{3(-2)-1(2)}{3-1}\right)$$

$$= \left(\frac{33+11}{2},\ \frac{-6-2}{2}\right) = (22, -4)$$

Let P $(22, -4) = (x_1, y_1)$

To find the equation of direct common tangents :

The equation of pair of tangents drawn from P to the circle S = 0 is $S_1^2 = S\,S_{11}$.

$$\therefore\ S_1 = x x_1 + y y_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x(22) + y(-4) + 11(x + 22) - 2(y - 4) - 100$$
$$= 22x - 4y + 11x + 242 - 2y + 8 - 100$$
$$= 33x - 6y + 150$$

$$S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$
$$= 22^2 + (-4)^2 + 22(22) - 4(-4) - 100$$
$$= 484 + 16 + 484 + 16 - 100$$
$$= 900$$

Now $S_1^2 = S\,S_{11}$

$\Rightarrow (33x - 6y + 150)^2 = [x^2 + y^2 + 22x - 4y - 100]\,900$

$\Rightarrow [3(11x - 2y + 50)]^2 = 900(x^2 + y^2 + 22x - 4y - 100)$

$\Rightarrow 9(11x - 2y + 50)^2 = 900(x^2 + y^2 + 22x - 4y - 100)$

$\Rightarrow 121x^2 + 4y^2 + 2500 - 44xy - 200y + 1100x$

$\qquad - 100x^2 - 100y^2 - 2200x + 400y + 10000 = 0$

$\Rightarrow 21x^2 - 96y^2 - 44xy - 1100x + 200y + 12500 = 0$

is the combined equation of the pair of tangents.

The seperate equation of the tangents

are $3x + 4y - 50 = 0$ and $7x - 24y - 250 = 0$

Second Method:

To find the equations of direct common tangents

The direct common tangents are drawn from P $(22, -4) = (x_1, y_1)$

Let m be the slope of the common tangent

Then the equation of tangent is, $y - y_1 = m(x - x_1)$

$\Rightarrow y + 4 = m(x - 22)$ .... (I)

$\Rightarrow mx - y - 22m - 4 = 0$ ... (1)

Now (1) is a tangent to the circle S = 0

$\Rightarrow$ radius = Length of perpendicular drawn from centre $C_1 = (-11, 2)$ to the line (1)

$$\Rightarrow 15 = \left|\frac{m(-11) - 2 - 22m - 4}{\sqrt{m^2 + 1^2}}\right|$$

$$\Rightarrow 15\sqrt{m^2 + 1} = |-33m - 6|$$

$$\Rightarrow 15\sqrt{m^2 + 1} = 3\,|-11m - 2|$$

$$\Rightarrow 5\sqrt{m^2 + 1} = -(11m + 2)$$

Squaring on both sides, we get

$$25(m^2 + 1) = (11m + 2)^2$$

$$\Rightarrow 25m^2 + 25 = 121m^2 + 4 + 44m$$

$$\Rightarrow 96m^2 + 44m - 21 = 0$$

$$\Rightarrow m = \frac{-44 \pm \sqrt{(44^2) - 4(96)(-21)}}{2 \times 96}$$

$$= \frac{-44 \pm \sqrt{10000}}{2 \times 96}$$

$$= \frac{-44 \pm 100}{2 \times 96} = \frac{-144}{2 \times 96} \text{ or } \frac{56}{2 \times 96}$$

$$= \frac{-3}{4} \text{ or } \frac{7}{24}$$

Substituting the values of 'm' in (I) we get the required direct common tangents as

$$y + 4 = \frac{-3}{4}(x - 22) \text{ and } y + 4 = \frac{7}{24}(x - 22)$$

$\Rightarrow 3x + 4y - 50 = 0$ and $7x - 24y - 250 = 0$

45. Find the transverse common tangents of circles $x^2 + y^2 - 4x - 10y + 28 = 0$ and $x^2 + y^2 + 4x - 6y + 4 = 0$

Sol.: Given circles are $S \equiv x^2 + y^2 - 4x - 10y + 28 = 0$

and $S' \equiv x^2 + y^2 + 4x - 6y + 4 = 0$.

For the circle S = 0, $2g = -4$, $2f = -10$, $c = 28$ $\Rightarrow g = -2, f = -5, c = 28$

$\therefore$ C_1 = centre = $(-g, -f) = (2, 5)$.

radius $r_1 = \sqrt{g^2 + f^2 - c} = \sqrt{4+25-28} = 1$

For the circle $S' = 0$, $2g = 4$, $2f = -6$, $c = 4$ $\Rightarrow g = 2, f = -3, c = 4$

Centre = $C_2 = (-2, 3)$, radius $r_2 = \sqrt{g^2 + f^2 - c} = \sqrt{4+9-4} = 3$

$$\text{Distance } \overline{C_1C_2} = \sqrt{(-2-2)^2 + (3-5)^2}$$

$$= \sqrt{16+4} = \sqrt{20} = 2\sqrt{5} \simeq 2\times(2.2)$$

$$\simeq 4.4$$

$r_1 + r_2 = 1+3 = 4 < C_1 C_2$

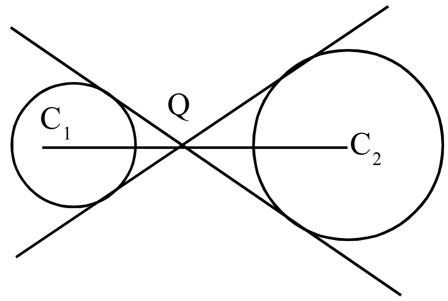
$\Rightarrow$ The two circles are non-intersecting circles.

The two transverse common tangents are drawn from the internal centre of similitude Q.

Q divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2 = 1 : 3$ internally

$$\therefore Q = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

$$= \left(\frac{1(-2)+3(2)}{1+3}, \frac{1(3)+3(5)}{1+3}\right)$$



$$= \left(\frac{-2+6}{4}, \frac{3+15}{4}\right)$$

$$= \left(1, \frac{9}{2}\right) = (x_1, y_1)$$

The transverse common tangents are drawn from Q. So, let the equation of tangent passing through Q with slope 'm' be, $y - y_1 = m(x - x_1)$

$$\Rightarrow y - \frac{9}{2} = m(x-1) \qquad \text{.... (I)}$$

$$\Rightarrow 2y - 9 = 2mx - 2m$$

$$\Rightarrow 2mx - 2y + 9 - 2m = 0 \qquad \text{.... (1)}$$

Now , (1) is a tangent to the circle S = 0

$\Rightarrow$ radius = perpendicular distance from the centre $C_1 = (2, 5)$ to the line (1)

$$\Rightarrow 1 = \left|\frac{2m(2) - 2(5) + 9 - 2m}{\sqrt{(2m)^2 + (-2)^2}}\right|$$

$\Rightarrow \sqrt{4m^2+4} = |2m-1|$

Squaring on both sides, we get

$4m^2 + 4 = (2m - 1)^2$

$\Rightarrow \quad 4m^2 + 4 = 4m^2 + 1 - 4m$

$\Rightarrow \quad 4m = -3$

$\Rightarrow \quad m = -\frac{3}{4}$

Substituting the value of 'm' in (I), we get

$y - \frac{9}{2} = \frac{-3}{4}(x-1)$

$\Rightarrow \quad \frac{2y-9}{2} = \frac{-3x+3}{4}$

$\Rightarrow \quad 4y - 18 = -3x + 3$

$\Rightarrow \quad 3x + 4y - 21 = 0$ is one of the transverse common tangent.

Since m^2 term is cancelled, slope of one of the transverse common tangents is not defined. So it is parallel to y - axis and passes through $Q\left(1, \frac{9}{2}\right)$

Any line parallel to y - axis is of the form $x = k$

Since it passes through $\left(1, \frac{9}{2}\right)$, $1 = k$.

$\therefore$ The equation of another transverse common tangent is $x = 1$ or $x - 1 = 0$

$\therefore$ The equations of the transverse common tangents are $x - 1 = 0$ and $3x + 4y - 21 = 0$

46. Show that the line $lx + my + n = 0$ is a normal to the circle S = 0 if and only if $gl + mf = n$.

Sol : The straight line $lx + my + n = 0$ is normal to the circle

$S = x^2 + y^2 + 2gx + 2fy + c = 0$

$\Leftrightarrow$ If the centre $(-g, -f)$ of the circle lies on $lx + my + n = 0$

$\Leftrightarrow \quad l(-g) + m(-f) + n = 0$

$\Leftrightarrow \quad lg + mf = n$

Unit **2**

System of Circles

Definition :

The Angle between two intersecting circles is defined as the angle between the tangents drawn at the point of intersection of the two circles.

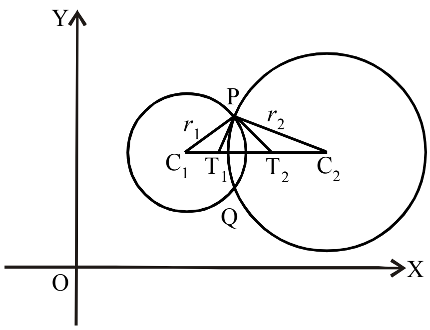
Note: If two circles $S = 0$ and $S' = 0$ intersect at the points P and Q, then the angle between the two circles at P and Q are equal.

Theorem : If C_1 and C_2 are the centres of two given intersecting circles, $d = C_1C_2$, r_1 and r_2 are the radii of these circles, θ is the angle between these circles, then prove that

$$\text{Cos}\,\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$$

Proof: Let 'P' be the point of intersection of two given circles. Let the tangents drawn to two circles at 'P' intersect the line joining the centres at T_1 and T_2.

Then $\angle T_1PT_2 = \theta$



$$\angle C_1PC_2 = \angle C_1PT_2 + \angle T_2PC_2$$
$$= 90^0 + 90^0 - \theta$$
$$= 180^0 - \theta$$

From ΔC_1PC_2,

according to cosine rule, we have

$C_1P = r_1$ is $\perp^r$ to tangent at P

$\Rightarrow \angle C_1PT_2 = 90^0$

Similarly,

$C_2P = r_2$ is $\perp^r$ to tangent at P

$\Rightarrow \angle C_2PT_1 = 90^0$

$$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2 - 2(C_1P)\,(C_2P)\cos\angle C_1PC_2$$

$$\Rightarrow d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(180^0 - \theta)$$

$$= r_1^2 + r_2^2 - 2r_1r_2[-\cos\theta]$$

$$\Rightarrow d^2 - r_1^2 - r_2^2 = 2r_1r_2\cos\theta$$

$$\Rightarrow \cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}.$$

$$\Rightarrow \angle T_1PT_2 + \angle T_2PC_2 = 90^0$$

$$\Rightarrow \theta + \angle T_2PC_2 = 90^0$$

$$\Rightarrow \angle T_2PC_2 = 90^0 - \theta$$

Note : Since Cos θ is independent of the coordinates of the point of intersection, the angle at Q is also equal to θ.

Theorem : If 'θ' is the angle between the intersecting circles $S = x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$, then

Show that $\text{Cos}\,\theta = \dfrac{c + c' - 2gg' - 2ff'}{2\times\sqrt{g^2+f^2-c}\sqrt{(g')^2+(f')^2-c'}}$

Proof :

Let C_1 and C_2 be the centres and r_1 and r_2 be the radii of the given circles $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ ...(2) respectively.

Then $C_1 = (-g, -f)$ $\quad C_2 = (-g', -f')$

$r_1 = \sqrt{g^2+f^2-c}$ $\quad r_2 = \sqrt{(g')^2+(f')^2-c'}$

$d = C_1C_2 = \sqrt{(g'-g)^2+(f'-f)^2}$ (∵ distance formula)

$$= (g')^2 + g^2 + (f')^2 + f^2 - 2gg' - 2ff'$$

$$\therefore d^2 - r_1^2 - r_2^2 = (g')^2 + g^2 + (f')^2 + f^2 - 2gg' - 2ff'$$

$$-(g^2+f^2-c) - [(g')^2+(f')^2-c']$$

$$= -2gg' - 2ff' + c + c'$$

$$= c' + c - 2gg' - 2ff'$$

If 'θ' is the angle between the intersecting circles (1) and (2) then

$$\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$$

$$= \frac{c + c' - 2gg' - 2ff}{2\times\sqrt{g^2+f^2-c}\times\sqrt{(g')^2+(f')^2-c'}}$$

Hence proved.

Definition : Two intersecting circles are said to be orthogonal, if the angle between them is a right angle, that is 90^0.

Condition for orthogonality

The condition for orthogonality of two intersecting circles

$S = x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$ is

$\boldsymbol{2gg' + 2ff' = c + c'}$

or $d^2 = r_1^2 + r_2^2$ where d = distance between the centres of the circles.

r_1, r_2 are their radii.

Theorem :

(i) If $S = 0$ and $S' = 0$ are two circles intersecting at two distinct points, then $S - S' = 0$ represents the common chord of these circles.

(ii) If $S = 0$ and $S' = 0$ are two circles touching each other, then $S - S' = 0$ is a common tangent at the point of contact.

Theorem : If $S = x^2 + y^2 + 2gx + 2fy + c = 0$ and $L = lx + my + n = 0$ are the equations of a circle and a straight line respectively intersecting each other, then the equation $S + \lambda L = 0$ represents a circle passing through the points of intersection of the circle $S = 0$ and the line $L = 0$, $\forall\ \lambda \in \mathbf{R}$.

If A and B are the points of intersection of the circle $S = 0$ and the line $L = 0$

Then the equation of any circle passing through A and B can be taken as $(S + \lambda L) = 0$

($\because$ There are many circles passing through A and B)

Theorem : If $S = x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$ are the equations of two intersecting circles, λ and μ are any real numbers such that $\lambda + \mu \neq 0$, then $\lambda S + \mu S' = 0$ or $S + kS' = 0$, $k \in \mathbf{R}$ represents a circle passing through the points of intersection of the circles $S = 0$ and $S' = 0$.

Note : If the circle $S = 0$ and $S' = 0$ intersect at A and B, then the equation of common chord $\overleftrightarrow{AB}$ is $S - S' = 0$

So the equation of any circle passing through A and B can also be taken as $S + \lambda(S - S') = 0$, where $\lambda \in \mathbf{R}$ taking the line $L = 0$ in $S + \lambda L = 0$ as $L = (S - S') = 0$

So the equation of any circle passing through A and B can be taken as $S + kS' = 0$, where $k \in \mathbf{R}$

or $\lambda S + \mu S' = 0$, where $\lambda, \mu \in \mathbf{R}$.

or $S + \lambda(S - S') = 0$, where $\lambda \in \mathbf{R}$.

Radical axis of two circles

Definition : The Radical axis of two circles is defined as the locus of a point which moves so that its powers with respect to the two circles are equal.

Theorem : If $S = x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$ are two non-concentric circles, then their radical axis is a straight line whose equation is $\boxed{S - S' = 0}$.

Note :

1) In the equation $S - S' = 0$, the circles $S = 0$ and $S' = 0$ should be in the standard form with coefficient of x^2 and coefficient of y^2, both equal to one.
2) For the concentric circles with distinct radii, the radical axis does not exist, since there is no point, whose powers w.r.t the two distinct concentric circles are equal.

Theorem : The radical axis of any two circles is perpendicular to the line joining their centres.

Theorem : The radical axis of two circles is

i) The 'common chord' when the two circles intersect at two distinct points.

ii) The 'common tangent' at the point of contact when the two circles touch each other.

Theorem : The radical axis of any two circles (whose commen tangent is not perpendicular to the line joining the centres) bisects the line joining the points of contact of common tangent to the circles.

Theorem : If the centres of any three circles are non-collinear, then the radical axes of each pair of the circles chosen from these three circles are concurrent.

The three radical axes, $S - S' = 0$, $S' - S'' = 0$ and $S - S'' = 0$ are concurrent at P.

This point 'P' is called as the radical centre.

Definition : (Learn the definition - very important)

The point of concurrence of the radical axes of each pair of the three circles whose centres are non-collinear is called as the **Radical centre**.

Note : The lengths of tangents drawn from the radical centre to these three circles are equal.

Theorem : If the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ cuts each of the two circles $S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0$ and $S'' = x^2 + y^2 + 2g''x + 2f''y + c'' = 0$ orthogonally, then the centre of $S = 0$ lies on the radical axis of $S' = 0$ and $S'' = 0$.

Theorem : Let $S' = 0$, $S'' = 0$ and $S''' = 0$ be three circles whose centres are non collinear and no two circles of these are intersecting, then the circle having

(i) radical centre of these circles as the centre of the circle and

(ii) length of tangent from the radical centre to any one of these circles as radius, cuts the given three circles orthogonally.

We apply this theorem in solving the problems.

PROBLEMS

1. Find the angle between the circles

$x^2 + y^2 + 4x - 14y + 28 = 0$ $x^2 + y^2 + 4x - 5 = 0$

Sol: Given circles are

$S = x^2 + y^2 + 4x - 14y + 28 = 0$ and $S' = x^2 + y^2 + 4x - 5 = 0$

$2g = 4,\ 2f = -14,\ c = 28$ $\quad$ $2g' = 4,\ 2f' = 0,\ c' = -5$

$\Rightarrow g = 2,\ 2f = -7,\ c = 28$ $\quad$ $\Rightarrow g' = 2,\ f' = 0,\ c' = -5$

Centre = $C_1 = (-g, -f) = (-2, 7)$ $\qquad$ $C_2 = (-g', -f') = (-2, 0)$

radius $= \sqrt{g^2+f^2-c}$ $\qquad$ $r_2 = \sqrt{(g')^2+(f')^2-c'} = \sqrt{4+0+5} = \sqrt{9}$

$r_1 = \sqrt{4+49-28}$ $\qquad$ $= 3$

$= \sqrt{25} = 5$

$d = C_1C_2 = \sqrt{(-2+2)^2+(7-0)^2} = 7$

If θ is the angle between the circles, then

$$\cos\theta = \frac{d^2-r_1^2-r_2^2}{2r_1r_2} = \frac{49-25-9}{2\times5\times3}$$

$$= \frac{15}{2\times5\times3} = \frac{1}{2} = \cos 60^0$$

$\Rightarrow \theta = 60^0$.

$\therefore$ The angle between the circles is 60^0.

Second Method :

Given circles are

$S = x^2+y^2+4x-14y+28=0$ and $S' = x^2+y^2+4x+0.y-5=0$

$2g = 4 \Rightarrow g = 2$ $\qquad$ $2g' = 4 \Rightarrow g' = 2$

$2f = -14 \Rightarrow f = -7$ $\qquad$ $2f' = 0 \Rightarrow f' = 0$

$c = 28 \Rightarrow c = 28$ $\qquad$ $c' = -5 \Rightarrow c' = -5$

If 'θ' is the angle between the curves, then

$$\cos\theta = \frac{c+c'-2gg'-2ff'}{2\times\sqrt{g^2+f^2-c}\times\sqrt{(g')^2+(f')^2-c'}}$$

$$= \frac{28-5-8-0}{2\times\sqrt{4+49-28}\times\sqrt{4+0+5}}$$

$$= \frac{15}{2\times5\times3} = \frac{1}{2}$$

$= \cos 60^0$.

$\therefore$ The angle between the two circles is 60^0.

2. If the angle between the circles $x^2+y^2-12x-6y+41=0$ and $x^2+y^2+kx+6y-59=0$ is 45^0, then find k.

Sol: Given circles are

$S = x^2+y^2+kx+6y-59=0$ and $S' = x^2+y^2-12x-6y+41=0$

$2g = k, 2f = 6, c = -59$ $\qquad$ $2g' = -12, 2f' = -6, c' = 41$

$\Rightarrow g = \frac{k}{2}, f = 3, c = -59$ $\qquad$ $\Rightarrow g' = -6, f' = -3, c' = 41.$

The angle between the circles is $45^0 \Rightarrow \theta = 45^0$.

$$\therefore \cos\theta = \frac{c+c'-2gg'-2ff'}{2\times\sqrt{g^2+f^2-c}\times\sqrt{(g')^2+(f')^2-c'}}$$

$$\Rightarrow \cos 45^0 = \frac{-59+41+6k+18}{2\sqrt{\frac{k^2}{4}+9+59}\times\sqrt{36+9-41}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{\overset{3}{\cancel{6}}k}{\cancel{2}\sqrt{\frac{k^2}{4}+68}\times\sqrt{4}} = \frac{3k}{\sqrt{\frac{k^2}{4}+68}\times\sqrt{4}}$$

Squaring on both sides we get

$$\frac{1}{2} = \frac{(3k)^2}{\left(\frac{k^2}{4}+68\right)\times 4}$$

$$\Rightarrow 2(9k^2) = \left(\frac{k^2}{4}+68\right)4$$

$$18k^2 = k^2 + 272$$

$$\Rightarrow \quad 17k^2 = 272 \qquad \Rightarrow k^2 = \frac{272}{17} = 16 \qquad \Rightarrow \boxed{k = \pm 4}$$

3. Show that the circles $x^2 + y^2 - 2x - 2y - 7 = 0$ and $3x^2 + 3y^2 - 8x + 29y = 0$ intersect each other orthogonally.

Sol: Given circles are

$S = x^2 + y^2 - 2x - 2y - 7 = 0$ and $S' = x^2 + y^2 - \frac{8}{3}x + \frac{29}{3}y + 0 = 0$

Always write the equations of the circles with coefficient of x^2 and coefficient of y^2 as one, ie, in the standard form

So, $3x^2 + 3y^2 - 8x + 29y = 0 \quad \Rightarrow \frac{3x^2}{3} + \frac{3y^2}{3} - \frac{8x}{3} + \frac{29y}{3} = 0$

$$\Rightarrow x^2 + y^2 - \frac{8x}{3} + \frac{29y}{3} = 0$$

$2g = -2,$	$\Rightarrow g = -1$	$2g' = \frac{-8}{3}$	$\Rightarrow g' = \frac{-4}{3}$
$2f = -2$	$\Rightarrow f = -1$	$2f' = \frac{29}{3}$	$\Rightarrow f' = \frac{29}{6}$
$c = -7$	$\Rightarrow c = -7$	$c' = 0$	$\Rightarrow c' = 0.$

So, $2gg' + 2ff' = 2(-1)\left(\frac{-4}{3}\right) + 2(-1)\left(\frac{29}{6}\right)$

$$= \frac{8}{3} - \frac{29}{3} = \frac{8-29}{3} = \frac{-21}{3} = -7$$

$c + c' = -7 + 0 = -7$

Since the condition $2gg' + 2ff' = c + c'$ is satisfied by the circles S = 0 and S$'$ = 0, they intersect each other orthogonally. Hence proved.

4. Find k, if the circles $x^2 + y^2 + 2by - k = 0$ and $x^2 + y^2 + 2ax + 8 = 0$ are orthogonal.

Sol: Given circles are $S = x^2 + y^2 + 2by - k = 0$

and $S' = x^2 + y^2 + 2ax + 8 = 0$

$2g = 0 \qquad 2g' = 2a$

$2f = 2b \qquad 2f' = 0$

$c = -k \qquad c' = 8.$

$\Rightarrow g = 0,\ f = b,\ c = -k,\ g' = a,\ f' = 0,\ c' = 8.$

It is given that the circles $S = 0$ and $S' = 0$ are orthogonal.

$\Rightarrow 2gg' + 2ff' = c + c'$

$\Rightarrow 2(0)(a) + 2(b)(0) = -k + 8$

$\Rightarrow 0 = -k + 8$

$\Rightarrow k = 8.$

5. Show that the angle between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = ax + ay$ is $\frac{3\pi}{4}$.

Sol: Given circles are $S = x^2 + y^2 - a^2 = 0$

and $S' = x^2 + y^2 - ax - ay = 0.$

$2g = 0, \Rightarrow g = 0 \qquad 2g' = -a \Rightarrow g' = \frac{-a}{2}$

$2f = 0 \Rightarrow f = 0 \qquad 2f' = -a \Rightarrow f' = \frac{-a}{2}$

$c = -a^2 \Rightarrow c = -a^2 \qquad c' = 0 \Rightarrow c' = 0.$

If 'θ' is the angle between the circles $S = 0$ and $S' = 0$ then

$$\cos\theta = \frac{c + c' - 2gg' - 2ff'}{2 \times \sqrt{g^2 + f^2 - c} \times \sqrt{(g')^2 + (f')^2 - c'}}$$

$$= \frac{-a^2 + 0 - 0 - 0}{2\sqrt{0 + 0 + a^2} \times \sqrt{\frac{a^2}{4} + \frac{a^2}{4} - 0}}$$

$$= \frac{-a^2}{2\sqrt{a^2} \times \sqrt{\frac{a^2}{2}}} \qquad \left[\because \frac{a^2}{4} + \frac{a^2}{4} = \frac{2a^2}{4} = \frac{a^2}{2}\right]$$

$$= \frac{-a^2}{2 \,.\, a \,.\, \frac{a}{\sqrt{2}}}$$

$$= \frac{-\sqrt{2}}{2} = \frac{-\sqrt{2}}{\sqrt{2} \times \sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}$$

$= \cos(180 - 45^0)$

$= \cos 135^0.$

$$\therefore \cos\theta = \cos \frac{3\pi}{4}.$$

$\therefore$ The angle between the circles $S = 0$ and $S' = 0$ is $\frac{3\pi}{4}$. Hence proved

Essay Problems

6. Find the equation of the circle which pass through (1, 1) and cuts orthogonally each of the circles $x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$

Sol: Let the circle required be $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

It passes through (1, 1) $\Rightarrow 1^2 + 1^2 + 2g(1) + 2f(1) + c = 0$

$\Rightarrow 2g + 2f + c + 2 = 0$ ...(2)

(1) is orthogonal to the circles $S' = x^2 + y^2 - 8x - 2y + 16 = 0$

$\Rightarrow 2gg' + 2ff' = c + c'$

$\Rightarrow 2g(-4) + 2f(-1) = c + 16$

$\Rightarrow -8g - 2f - c - 16 = 0$ ...(3)

$$\left(\begin{array}{l} \because 2g' = -8 \Rightarrow g' = -4 \\ 2f' = -2 \Rightarrow f' = -1 \\ c' = 16 \Rightarrow c' = 16 \end{array}\right)$$

Again (1) is orthogonal to $x^2 + y^2 - 4x - 4y - 1 = 0$

$\Rightarrow 2gg' + 2ff' = c + c'$

$\Rightarrow 2g(-2) + 2f(-2) = c - 1$

$\Rightarrow -4g - 4f - c + 1 = 0$ ...(4)

$$\left(\begin{array}{l} \because 2g' = -4 \Rightarrow g' = -2 \\ 2f' = -4 \Rightarrow f' = -2 \\ c' = -1. \end{array}\right)$$

Solving (2), (3) and (4) :-

(2) $\Rightarrow 2g + 2f + c + 2 = 0$

(3) $\Rightarrow -8g - 2f - c - 16 = 0$

$-6g - 14 = 0$

(3) $-8g - 2f - c - 16 = 0$

(4) $-4g - 4f - c + 1 = 0$

$\quad + \quad + \quad + \quad -$

$-4g + 2f - 17 = 0$

$\Rightarrow g = \frac{-14}{6} = \frac{-7}{3}$

$\Rightarrow -4\left(-\frac{7}{3}\right) + 2f - 17 = 0$

$\Rightarrow \frac{28}{3} + 2f - 17 = 0$

$\Rightarrow 2f = \frac{23}{3} \quad \Rightarrow \boxed{f = \frac{23}{6}}$

Substituting the values of 'g' and 'f' in (2),

we get $2\left(\frac{-7}{3}\right) + 2\left(\frac{23}{6}\right) + c + 2 = 0$

$\Rightarrow \frac{-14}{3} + \frac{23}{3} + c + 2 = 0$

$\Rightarrow c = \frac{14}{3} - \frac{23}{3} - 2 = \frac{14-23-6}{3} = \frac{-15}{3} = -5.$

Substituting the values of g, f, c in (1) we get the required circle as

$$x^2 + y^2 + 2\left(\frac{-7}{3}\right)x + 2\left(\frac{23}{6}\right)y - 5 = 0$$

$\Rightarrow 3x^2 + 3y^2 - 14x + 23y - 15 = 0$

7. Find the equation of the circle which is orthogonal to each of the following 3 circles.
$x^2 + y^2 + 2x + 17y + 4 = 0$, $x^2 + y^2 + 7x + 6y + 11 = 0$. and $x^2 + y^2 - x + 22y + 3 = 0$.

Sol: Given circles are

$S' = x^2 + y^2 + 2x + 17y + 4 = 0$ ...(1)

$S'' = x^2 + y^2 + 7x + 6y + 11 = 0$ ...(2)

$S''' = x^2 + y^2 - x + 22y + 3 = 0$ ...(3)

let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ ...(4) be the required circle orthogonal to (1), (2) and (3)

Then (1) and (4) are orthogonal

$\Rightarrow 2gg' + 2ff' = c + c'$

$\Rightarrow 2g(1) + 2f\left(\frac{17}{2}\right) = c + 4$

$\Rightarrow 2g + 17f = c + 4$ ...(5)

$\left[\begin{array}{l} \because 2g' = 2 \Rightarrow g' = 1 \\ 2f' = 17 \Rightarrow f' = \frac{17}{2} \\ c' = 4 \end{array}\right]$

Again (2) and (4) are orthogonal

$\Rightarrow 2gg'' + 2ff'' = c + c''$

$\Rightarrow 2g\left(\frac{7}{2}\right) + 2f\left(\frac{6}{2}\right) = c + 11$

$\Rightarrow 7g + 6f = c + 11$ ...(6)

$\left[\begin{array}{l} \because 2g'' = 7 \Rightarrow g'' = \frac{7}{2} \\ 2f'' = 6 \Rightarrow f'' = \frac{6}{2} \\ c'' = 11. \end{array}\right]$

Again (3) and (4) are orthogonal

$\Rightarrow 2g\,g''' + 2f\,f''' = c + c''$

$\Rightarrow 2g\left(-\frac{1}{2}\right) + 2f(11) = c + 3$

$\Rightarrow -g + 22f = c + 3$ ...(7)

$$\left(\begin{array}{lll} \because\ 2g''' = -1 & \Rightarrow & g''' = -\frac{1}{2} \\ 2f''' = 22 & \Rightarrow & f''' = 11 \\ c''' = 3 & & \end{array}\right)$$

Solving (5), (6) and (7) we get

(5) $\Rightarrow 2g + 17f = \cancel{c} + 4$

(6) $\Rightarrow 7g + 6f = \cancel{c} + 11$

$\quad - \quad - \qquad - \quad -$

$-5g + 11f = -7$ ...(8)

(6) $\Rightarrow 7g + 6f = \cancel{c} + 11$

(7) $\Rightarrow -g + 22f = \cancel{c} + 3$

$\quad + \quad - \qquad - \quad -$

$8g - 16f = 8$ ...(9)

Solving (8) and (9), we get

$8(-5g + 11f = -7)$

$5(\ 8g - 16f = 8)$

$-40g + 88f = -56$

$40g - 80f = 40$

$8f = -16$

$f = \frac{-16}{8}$

$\boxed{f = -2}$

Substituting $f = -2$ in (8)

we get

$-5g + 11(-2) = -7$

$\Rightarrow -5g = -7 + 22 = 15$

$\Rightarrow g = \frac{15}{-5} = -3.$

$\boxed{g = -3}$

Substituting the values of 'g' and 'f' in (7) we get

$-g + 22f = c + 3$

$\Rightarrow \cancel{3} + 22(-2) = c + \cancel{3}$

$\Rightarrow \boxed{c = -44}$

Substituting the values of 'g' 'f' and 'c' in (4), we get the required circles as

$x^2 + y^2 + 2(-3)x + 2(-2)y - 44 = 0$

$\Rightarrow x^2 + y^2 - 6x - 4y - 44 = 0$

8. Find the equation of the circle passing through the origin, having its centre on the line $x + y = 4$ and intersecting the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.

Sol: Let the required circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

It passes through origin $\Rightarrow c = 0$ ...(2)

Its centre $(-g, -f)$ lies on the line $x + y = 4$

$\Rightarrow (-g) + (-f) = 4 \Rightarrow -g - f = 4$ ...(3)

(1) intersects the circle $S' = x^2 + y^2 - 4x + 2y + 4 = 0$ ...(4)

orthogonally

$\Rightarrow 2gg' + 2ff' = c + c'$ $\left[\because 2g' = -4 \Rightarrow g' = -2,\ 2f' = 2 \Rightarrow f' = 1,\ c' = 4\right]$

$\Rightarrow 2g(-2) + 2f(1) = c + 4$

$\Rightarrow -4g + 2f = 0 + 4 \quad [\because c = 0]$

$\Rightarrow 2(-2g + f) = 4$

$\Rightarrow -2g + f = 2 \quad ...(5)$

Solving (3) & (5) we get

$-g - f = 4$

$\underline{-2g + f = 2}$

$-3g = 6$

$\Rightarrow g = \frac{6}{-3}$

$\Rightarrow \boxed{g = -2}$

Substituting $g = -2$ in (3) we get

$-(-2) - f = 4$

$\Rightarrow -f = 4 - 2 = 2$

$\Rightarrow \boxed{f = -2}$

Substituting the values of g, f and c in (1) we get

$$x^2 + y^2 + 2(-2)x + 2(-2)y = 0$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 4y = 0.$$

9. Find the equation of the circle which passes through the points (2, 0), (0, 2) and orthogonal to the circle $2x^2 + 2y^2 + 5x - 6y + 4 = 0$.

Sol: Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1) be the required circle.

It passes through (2, 0)

$\Rightarrow 2^2 + 0^2 + 2g(2) + 2f(0) + c = 0$

$\Rightarrow 4g + c + 4 = 0 \quad ...(2)$

Circle (1) passes through (0, 2) $\Rightarrow 0^2 + 2^2 + 2g(0) + 2f(2) + c = 0$

$\Rightarrow 4f + c + 4 = 0 \quad ...(3)$

Circle (1) is orthogonal to $2x^2 + 2y^2 + 5x - 6y + 4 = 0$

that is $x^2 + y^2 + \frac{5}{2}x - \frac{6}{2}y + \frac{4}{2} = 0$

$\Rightarrow 2gg' + 2ff' = c + c'$ $\left[\because 2g' = \frac{5}{2} \Rightarrow g' = \frac{5}{4},\ 2f' = \frac{-6}{2} \Rightarrow f' = \frac{-3}{2},\ c' = \frac{4}{2} = 2.\right]$

$\Rightarrow 2g\left(\frac{5}{4}\right) + 2f\left(\frac{-3}{2}\right) = c + 2$

$\Rightarrow \frac{5g}{2} - 3f = c + 2 \quad ...(4)$

Solving (2), (3) and (4) we get

(2) $\Rightarrow$ $4g + c + 4 = 0$	(2) $\Rightarrow$ $4g + c + 4 = 0$
(3) $\Rightarrow$ $4f + c + 4 = 0$	(4) $\Rightarrow$ $\frac{5g}{2} - 3f - c - 2 = 0$
$-\quad -\quad -$	
$4g - 4f = 0$	$4g + \frac{5g}{2} - 3f + 2 = 0$
$\Rightarrow 4g = 4f$	But $g = f$
$\Rightarrow g = f$	$\Rightarrow 4g + \frac{5g}{2} - 3g + 2 = 0$
	$\Rightarrow \frac{8g + 5g - 6g + 4}{2} = 0$
	$\Rightarrow 7g + 4 = 0$
	$\Rightarrow g = \frac{-4}{7} = f.$

From (2) $\because c = -4g - 4$

$$= -4\left(\frac{-4}{7}\right) - 4 = \frac{16}{7} - 4 = \frac{16 - 28}{7} = \frac{-12}{7}$$

Substituting in (1) we get the required circle as

$$x^2 + y^2 + \left(\frac{-8}{7}\right)x + \left(\frac{-8}{7}\right)y - \frac{12}{7} = 0$$

$\Rightarrow$ $7x^2 + 7y^2 - 8x - 8y - 12 = 0$ Ans.

10. Find the equation of the circle which cuts orthogonally the circle $x^2 + y^2 - 4x + 2y - 7 = 0$ and having the centre at (2, 3).

Sol: Let the required circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1)

Its centre is (2, 3) $\Rightarrow (-g, -f) = (2, 3)$

$\Rightarrow -g = 2, -f = 3$

$\Rightarrow \boxed{g = -2}, \boxed{f = -3}$

The circle (1) is orthogonal to $S' = x^2 + y^2 - 4x + 2y - 7 = 0$

$\Rightarrow 2gg' + 2ff' = c + c'$ $\quad [\because 2g' = -4, 2f' = 2, c' = -7]$

$\Rightarrow 2(-2)(-2) + 2(-3)(1) = c - 7$

$\Rightarrow 8 - 6 = c - 7$

$\Rightarrow \boxed{c = 9}$

substituting in (1) we get the required circle as

$$x^2 + y^2 + 2(-2)x + 2(-3)y + 9 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 6y + 9 = 0.$$

11. Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 8x - 6y + 21 = 0$ and $x^2 + y^2 - 2x - 15 = 0$ and (1, 2)

Sol: Given circles are

$$S = x^2 + y^2 - 8x - 6y + 21 = 0$$

and $\quad S' = x^2 + y^2 - 2x - 15 = 0$

Now $S - S' = x^2 + y^2 - 8x - 6y + 21 - x^2 - y^2 + 2x + 15$

$$= -6x - 6y + 36$$

We know that, the equation of any circle passing through the points of intersection of the circle S = 0 and S′ = 0 is $S + \lambda(S - S') = 0, \qquad \lambda \in \mathbf{R}$.

So let the required circle be $S + \lambda(S - S') = 0$

$\Rightarrow \quad x^2 + y^2 - 8x - 6y + 21 + \lambda(-6x - 6y + 36) = 0 \qquad ...(1)$

It passes through (1, 2)

$\Rightarrow \quad 1^2 + 2^2 - 8(1) - 6(2) + 21 + \lambda(-6(1) - 6(2) + 36) = 0$

$\Rightarrow \quad 1 + 4 - 8 - 12 + 21 + \lambda(-6 - 12 + 36) = 0$

$\Rightarrow \quad 6 + \lambda(18) = 0$

$\Rightarrow \lambda = \dfrac{-6}{18} = \dfrac{-1}{3}$

Substituting $\lambda = -\dfrac{1}{3}$ in (1), we get the required circle as

$$x^2 + y^2 - 8x - 6y + 21 + \frac{-1}{3}(-6x - 6y + 36) = 0$$

$\Rightarrow \quad x^2 + y^2 - 8x - 6y + 21 + 2x + 2y - 12 = 0$

$\Rightarrow \quad x^2 + y^2 - 6x - 4y + 9 = 0$

12. If $x + y = 3$ is the equation of the chord AB of the circle $x^2 + y^2 - 2x + 4y - 8 = 0$, then find the equation of the circle having $\overline{AB}$ as diameter.

Sol: Let the given circle $S = x^2 + y^2 - 2x + 4y - 8 = 0$ and the line $L = x + y - 3 = 0$ intersect at A and B.

Then the equation of any circle passing through A and B is $S + \lambda L = 0$

$\Rightarrow \quad x^2 + y^2 - 2x + 4y - 8 + \lambda(x + y - 3) = 0 \qquad ...(I)$

$\Rightarrow \quad x^2 + y^2 + (\lambda - 2)x + (4 + \lambda)y - 8 - 3\lambda = 0 \qquad ...(1)$

If (1) itself is the required circle with $\overline{AB}$ as diameter then its centre

$C = \left(\dfrac{-(\lambda - 2)}{2}, \dfrac{-(4 + \lambda)}{2}\right)$ lies on the line L = 0.

Substituting $C = \left(\dfrac{-(\lambda - 2)}{2}, \dfrac{-(4 + \lambda)}{2}\right)$ in L = 0 we get, $\frac{-(\lambda-2)}{2}+\frac{-(4+\lambda)}{2}-3=0$

$\Rightarrow \frac{-\lambda+2-4-\lambda-6}{2}=0$

$\Rightarrow -2\lambda-8=0$

$\Rightarrow -2\lambda=8 \qquad \Rightarrow \lambda=\frac{8}{-2} \qquad \Rightarrow \boxed{\lambda=-4}$

Substituting $\lambda=-4$ in (1), we get the required circle as

$x^2+y^2-2x+4y-8-4(x+y-3)=0$

$\Rightarrow \quad x^2+y^2-6x+4=0.$

13. Find the equation of the radical axis of the circles $2x^2+2y^2+3x+6y-5=0$ and $3x^2+3y^2-7x+8y-11=0$

Sol: Let the circles in the standard form be

$S=\frac{2x^2}{2}+\frac{2y^2}{2}+\frac{3x}{2}+\frac{6y}{2}-\frac{5}{2}=0$

$\Rightarrow S=x^2+y^2+\frac{3}{2}x+3y-\frac{5}{2}=0$

and $S'=x^2+y^2-\frac{7}{3}x+\frac{8}{3}y-\frac{11}{3}=0$

The radical axis of $S=0$ and $S'=0$ is $S-S'=0$

$\Rightarrow \cancel{x^2}+\cancel{y^2}+\frac{3}{2}x+3y-\frac{5}{2}-\cancel{x^2}-\cancel{y^2}+\frac{7}{3}x-\frac{8}{3}y+\frac{11}{3}=0$

$\Rightarrow \frac{3}{2}x+3y-\frac{5}{2}+\frac{7}{3}x-\frac{8}{3}y+\frac{11}{3}=0$

$\Rightarrow \frac{9x+18y-15+14x-16y+22}{6}=0$

$\Rightarrow 23x+2y+7=0$

14. Find the radical centre of the circles

$x^2+y^2-2x+6y=0$, $x^2+y^2-4x-2y+6=0$ and $x^2+y^2-12x+2y+3=0$

Sol: Let the given circles be

$S=x^2+y^2-2x+6y=0,$

$S'=x^2+y^2-4x-2y+6=0$

$S''=x^2+y^2-12x+2y+3=0$

The radical axis of $S=0$ and $S'=0$ is $S-S'=0$

$\Rightarrow \cancel{x^2}+\cancel{y^2}-2x+6y-\cancel{x^2}-\cancel{y^2}+4x+2y-6=0$

$\Rightarrow 2x+8y-6=0$

$\Rightarrow \quad x+4y-3=0 \qquad \ldots(1)$

Similarly the radical axis of $S'=0$ and $S''=0$ is $S'-S''=0$

$\Rightarrow\ x^2+y^2-4x-2y+6-x^2-y^2+12x-2y-3=0$

$\Rightarrow\ 8x-4y+3=0$ ...(2)

Solving (1) and (2) we get the radical centre.

$x+4y-3=0$

$\underline{8x-4y+3=0}$

$9x=0$

$\Rightarrow x=0$

substituting $x=0$ in (1), we get $y=\dfrac{3}{4}$.

$\therefore$ The radical centre is $\left(0,\ \dfrac{3}{4}\right)$.

Note : To find Radical centre solve any two of the following radical axes : $(S-S')=0$, $(S'-S'')=0$, $(S-S'')=0$.

15. Find the equation of the common chord and also its length of the two circles.

$S=x^2+y^2+3x+5y+4=0$ and $S'=x^2+y^2+5x+3y+4=0$

Sol: The given circles are

$$S=x^2+y^2+3x+5y+4=0$$

and $$S'=x^2+y^2+5x+3y+4=0$$

For circle $S=0$

centre $C_1=\left(\dfrac{-3}{2},\dfrac{-5}{2}\right)$

$r_1=\sqrt{\dfrac{9}{4}+\dfrac{25}{4}-4}$

$=\sqrt{\dfrac{9+25-16}{4}}=\sqrt{\dfrac{18}{4}}$

$=\sqrt{\dfrac{9}{2}}=\dfrac{3}{\sqrt{2}}$

$r_1+r_2=\dfrac{6}{\sqrt{2}}$

$r_1+r_2=\dfrac{6}{\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=3\sqrt{2}$

For circle $S'=0$

$C_2=\left(\dfrac{-5}{2},\dfrac{-3}{2}\right)$

$r=\sqrt{\dfrac{25}{4}+\dfrac{9}{4}-4}$

$=\dfrac{3}{\sqrt{2}}$

Distance $C_1C_2=\sqrt{\left(\dfrac{-5}{2}+\dfrac{3}{2}\right)^2+\left(\dfrac{-3}{2}+\dfrac{5}{2}\right)^2}$

$=\sqrt{\left(\dfrac{-2}{2}\right)^2+\left(\dfrac{2}{2}\right)^2}=\sqrt{2}.$

$3\sqrt{2}>\sqrt{2}$ or $\sqrt{2}<3\sqrt{2}$

$\Rightarrow\ C_1C_2<r_1+r_2$

$r_1 - r_2 = 0$

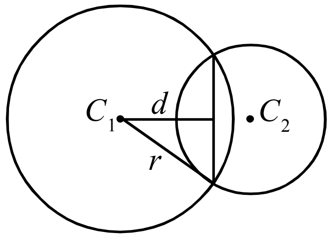
$|r_1 - r_2| < C_1C_2 < |r_1 + r_2|$

$\Rightarrow$ The circles intersect. So the radical axis $S - S' = 0$ is the common chord.

$\Rightarrow \; \cancel{x^2} + \cancel{y^2} + 3x + 5y + \cancel{4} - \cancel{x^2} - \cancel{y^2} - 5x - 3y - \cancel{4} = 0$

$-2x + 2y = 0$ or $x - y = 0$ is the radical axis.

$\therefore$ The equation of common chord is $L = x - y = 0$ ...(1)



The length of common chord is $2\sqrt{r^2 - d^2}$

where 'r' is the radius of the circle $S = 0$

& d is the perpendicular distance from C_1 to the line (1)

$$d = \frac{\left|\frac{-3}{2} + \frac{5}{2}\right|}{\sqrt{1^2 + (-1)^2}} \qquad \left(\begin{array}{l} \because \text{ formula is } \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ \text{where } (x_1, y_1) = C_1 = \left(\dfrac{-3}{2}, \dfrac{-5}{2}\right) \\ ax + by + c \text{ is } x - y. \Rightarrow a = 1, b = -1, c = 0 \end{array} \right)$$

$$= \frac{1}{\sqrt{2}}$$

$\therefore$ Length of common chord $= 2\sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2} = 2\sqrt{\frac{9}{2} - \frac{1}{2}}$

$= 2 \times 2 = 4$ units.

16. Show that the circles $S = x^2 + y^2 - 2x - 4y - 20 = 0$ and $S' = x^2 + y^2 + 6x + 2y - 90 = 0$ touch each other internally. Find their point of contact and the equation of common tangent.

Sol: Given circles are $S = x^2 + y^2 - 2x - 4y - 20 = 0$ and $S' = x^2 + y^2 + 6x + 2y - 90 = 0$

For circle $S = 0$,

centre $= C_1 = (1, 2)$

radius $= r_1 = \sqrt{1 + 4 + 20}$

$= \sqrt{25} = 5$

For circle $S' = 0$

centre $= C_2 = (-3, -1)$

radius $= r_2 = \sqrt{9 + 1 + 90}$

$= \sqrt{100} = 10$

Distance $C_1C_2 = \sqrt{(-3-1)^2 + (-1-2)^2}$

$= \sqrt{16 + 9} = \sqrt{25} = 5$

we observe that $C_1C_2 = |r_1 - r_2|$ $[\because 5 = |5 - 10|]$

So the two circles touch each other internally.

The equation of common tangent at the point of contact is the radical axis $S - S' = 0$

$\Rightarrow x^2 + y^2 - 2x - 4y - 20 - x^2 - y^2 - 6x - 2y + 90 = 0$

$\Rightarrow -8x - 6y + 70 = 0$

$\Rightarrow -2(4x + 3y - 35) = 0$

$\Rightarrow 4x + 3y - 35 = 0$

The point of contact P, is the external centre of similitude which divides C_1C_2 in the ratio $r_1 : r_2 = 5 : 10 = 1 : 2$ externally.

$$\therefore \; P = \frac{mx_2 - nx_1}{m-n}, \; \frac{my_2 - ny_1}{m-n}$$

$$= \left(\frac{-3-2}{1-2}, \frac{-1-4}{1-2}\right)$$

$$= \left(\frac{-5}{-1}, \frac{-5}{-1}\right)$$

$$= (5, 5)$$

Note : The point of contact is also the foot of the perpendicular drawn from C_1 or C_2 to the common tangent $4x + 3y - 35 = 0$.

Let $P(h, k)$, $C_1 = (x_1, y_1) = (1, 2)$ tangent : $4x + 3y - 35 = 0$ is $ax + by + c = 0$

Then

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2} \qquad [\because a = 4 \;\; b = 3 \;\; c = -35]$$

$$\Rightarrow \frac{h-1}{4} = \frac{k-2}{3} = \frac{-(4+6-35)}{4^2+3^2}$$

$$= \frac{25}{25} = 1$$

$$\left.\begin{aligned} &\Rightarrow \frac{h-1}{4} = 1, \; \frac{k-2}{3} = 1 \\ &\Rightarrow h - 1 = 4, \; k - 2 = 3 \end{aligned}\right\} \quad h = 5, \;\; k = 5.$$

$\therefore$ The point of contact is $(h, k) = (5, 5)$.

17. Find the equation of the circle which cuts the circle $x^2 + y^2 + 2x + 4y + 1 = 0$, $2x^2 + 2y^2 + 6x + 8y - 3 = 0$ and $x^2 + y^2 - 2x + 6y - 3 = 0$ orthogonally.

Sol: Let the given circles be

$S = x^2 + y^2 + 2x + 4y + 1 = 0$ ...(1)

$$S' = x^2 + y^2 + \frac{6}{2}x + \frac{8}{2}y - \frac{3}{2} = 0$$

$$\Rightarrow \; S' = x^2 + y^2 + 3x + 4y - \frac{3}{2} = 0 \qquad \text{...(2)}$$

and $\Rightarrow \; S'' = x^2 + y^2 - 2x + 6y - 3 = 0$ ...(3)

The radical axis of (1) and (2) is $S - S' = 0$

$$\Rightarrow \; \cancel{x^2} + \cancel{y^2} + 2x + \cancel{4y} + 1 - \cancel{x^2} - \cancel{y^2} - 3x - \cancel{4y} + \frac{3}{2} = 0$$

$\Rightarrow -x+1+\dfrac{3}{2} = 0$

$\Rightarrow -x+\dfrac{5}{2} = 0$ ...(4)

The radical axis of (1) and (3) is $S - S'' = 0$

$\Rightarrow \cancel{x^2} + y^2 + 2x + 4y + 1 - \cancel{x^2} - y^2 + 2x - 6y + 3 = 0$

$\Rightarrow 4x - 2y + 4 = 0$

$\Rightarrow 2(2x - y + 2) = 0$

$\Rightarrow 2x - y + 2 = 0$ ...(5)

Solving (4) and (5) we ge the radical centre

(4) $\Rightarrow -x+\dfrac{5}{2} = 0 \Rightarrow x = \dfrac{5}{2}$

Substituing in (5) we get

$2\left(\dfrac{5}{2}\right) - y + 2 = 0$

$\Rightarrow 5 - y + 2 = 0$

$\Rightarrow y = 7$

$\therefore$ The radical centre is $\left(\dfrac{5}{2}, 7\right) = (x_1, y_1)$

Now length of tangent from $\left(\dfrac{5}{2}, 7\right)$ to circle $S = 0$ is

$\sqrt{S_{11}} = \sqrt{x_1^2 + y_1^2 + 2x_1 + 4y_1 + 1}$

$= \sqrt{\left(\dfrac{5}{2}\right)^2 + 7^2 + 2\left(\dfrac{5}{2}\right) + 4(7) + 1}$

$= \sqrt{\dfrac{25}{4} + 49 + 5 + 28 + 1}$

$= \sqrt{\dfrac{25}{4} + 83}$

$= \sqrt{\dfrac{25+332}{4}}$

$= \sqrt{\dfrac{357}{4}}$

$\therefore$ The circle orthogonal to (1), (2) and (3) is

$$(x - x_1)^2 + (y - y_1)^2 = (\sqrt{S_{11}})^2$$

$$\Rightarrow \left(x - \frac{5}{2}\right)^2 + (y-7)^2 = \frac{357}{4}$$

$$\Rightarrow x^2 + y^2 + \frac{25}{4} + 49 - 5x - 14y = \frac{357}{4}$$

$$\Rightarrow x^2 + y^2 - 5x - 14y + \frac{25}{4} + 49 - \frac{357}{4} = 0$$

$\Rightarrow x^2 + y^2 - 5x - 14y - 34 = 0$ is the required circle.

Second Method

Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$

be the circle orthogonal to

$S' = x^2 + y^2 + 2x + 4y + 1 = 0$ $\qquad 2g' = 2,\ 2f' = 4,\ c' = 1$

$S'' = x^2 + y^2 + 3x + 4y - \frac{3}{2} = 0$ $\qquad 2g'' = 3,\ 2f'' = 4,\ c'' = \frac{-3}{2}$

$S''' = x^2 + y^2 - 2x + 6y - 3 = 0$ $\qquad 2g''' = -2,\ 2f''' = 6,\ c''' = -3$

$S = 0$ and $S' = 0$ are orthogonal

$\Rightarrow 2gg' + 2ff' = c + c'$

$\Rightarrow 2g(1) + 2f(2) = c + 1 \qquad \Rightarrow 2g + 4f = c + 1 \qquad ...(1)$

Again

$S = 0$ and $S'' = 0$ are orthogonal

$\Rightarrow 2gg'' + 2ff'' = c + c''$

$\Rightarrow \cancel{2}g\left(\frac{3}{\cancel{2}}\right) + 2f(2) = c - \frac{3}{2} \qquad \Rightarrow 3g + 4f = c - \frac{3}{2} \qquad ...(2)$

Again $S = 0$ and $S''' = 0$ are orthogonal

$\Rightarrow 2gg''' + 2ff''' = c + c'''$

$\Rightarrow 2g(-1) + 2f(3) = c - 3 \qquad \Rightarrow -2g + 6f = c - 3 \qquad ...(3)$

Solving (1), (2) and (3), we get

$(1) \Rightarrow 2g + 4f = c + 1$ $\qquad (1) \Rightarrow 2g + 4f = c + 1$

$(2) \Rightarrow 3g + 4f = c - \frac{3}{2}$ $\qquad (3) \Rightarrow -2g + 6f = c - 3$

$\quad - \quad - \quad - \ +$ $\qquad + \quad - \quad - \ +$

$-g = 1 + \frac{3}{2}$ $\qquad 4g - 2f = 4$

$\Rightarrow \boxed{g = \dfrac{-5}{2}}$

$\Rightarrow 4\left(\dfrac{-5}{2}\right) - 2f = 4$

$\Rightarrow -2f = 4 + 10$

$f = \dfrac{14}{-2} = -7.$

Substituting $g = \dfrac{-5}{2}$ & $f = -7$ in (1), we get

$$2\left(\frac{-5}{2}\right) + 4(-7) = c + 1$$

$\Rightarrow -5 - 28 - 1 = c \Rightarrow c = -34$

Substituting the values of 'g', 'f' and 'c' in S = 0, we get the required circle as

$x^2 + y^2 - 5x - 14y - 34 = 0.$

18. Show that the common chord of the circles $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 8x - 6y + 23 = 0$ is the daimeter of the second circle and also find its length.

Sol : Given circles are

$S = x^2 + y^2 - 6x - 4y + 9 = 0$ $\quad 2g = -6,\ 2f = -4,\ c = 9$

and $S' = x^2 + y^2 - 8x - 6y + 23 = 0$ $\quad 2g' = -8,\ 2f' = -6,\ c' = 23$

For circle S = 0 $\quad$ For circle $S' = 0$

Centre = $C_1 = (3,2)$ $\quad$ Centre = $C_2 = (4,3)$

raidus = $r_1 = \sqrt{9+4-9} = 2$ $\quad$ radius = $r_2 = \sqrt{16+9-23} = \sqrt{2}$

Distance $C_1C_2 = \sqrt{(4-3)^2 + (3-2)^2} = \sqrt{2} < 2 + \sqrt{2}.$

$|r_1 - r_2| < C_1C_2 < r_1 + r_2$ $\quad [\because 2 - \sqrt{2} = 2 - 1.414 = 0.586]$

$\Rightarrow$ The circles are intersecting circles.

The common chord is the radical axis $S - S' = 0$

$\Rightarrow x^2 + y^2 - 6x - 4y + 9 - x^2 - y^2 + 8x + 6y - 23 = 0$

$\Rightarrow 2x + 2y - 14 = 0$

$\Rightarrow x + y - 7 = 0$ ... (1) is the common chord

To show that it is the diameter of second circle $S' = 0$:-

Centre of $S' = 0$ is (4,3)

Substituting (4, 3) in 1, we get $4 + 3 - 7 = 0$

$\Rightarrow$ The centre of the circle $S' = 0$ lies on the radical axis that is, the common chord $\overline{AB}$ of the circles S = 0 & $S' = 0$.

$\therefore$ The common chord is the diameter of the second circle $S' = 0$.

Hence proved.

$\therefore$ Length of common chord

= length of the diameter of circle $S' = 0$

$= 2 \times$ radius of circle $S' = 2\sqrt{2}$ units

OR

Length of common chord = $2\sqrt{r^2 - d^2}$

where r = radius of circle $S = 0$

d = length of the perpendicular from the center $C_1 = (3,2)$ to the chord $x + y - 7 = 0$

$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \text{ (formula)}$$

$$= \frac{|3+2-7|}{\sqrt{1^2+1^2}} = \frac{2}{\sqrt{2}}$$

$$= \frac{\sqrt{2}\ .\ \sqrt{2}}{\sqrt{2}} = \sqrt{2}$$

Length of common chord $= 2\sqrt{r^2 - d^2}$

$$= 2 \times \sqrt{2^2 - (\sqrt{2})^2}$$

$$= 2 \times \sqrt{4-2}$$

$$= 2\sqrt{2} \text{ units.}$$

f

19. Find the equation of the circle whose diameter is the common chord of the circles $S \equiv x^2 + y^2 + 2x + 3y + 1 = 0$ and $S' \equiv x^2 + y^2 + 4x + 3y + 2 = 0$

Sol : Given circles are

$S = x^2 + y^2 + 2x + 3y + 1 = 0$

and $S' = x^2 + y^2 + 4x + 3y + 2 = 0$

For circle $S = 0$,

Centre $= C_1 = \left(-1, \frac{-3}{2}\right)$

radius $= r_1 = \sqrt{1 + \frac{9}{4} - 1}$

$r_1 = \sqrt{\frac{9}{4}} = \frac{3}{2}.$

For circle $S' = 0$,

Centre $= C_2 = \left(-2, \frac{-3}{2}\right)$

radius $= r_2 = \sqrt{4 + \frac{9}{4} - 2}$

$= \sqrt{2 + \frac{9}{4}} = \frac{\sqrt{17}}{2}$

$r_2 = \frac{4.12}{2} = 2.06.$

Distance $C_1C_2 = \sqrt{(-2+1)^2 + \left(\frac{-3}{2}+\frac{3}{2}\right)^2}$

$= 1.$

$1 < \left(\frac{3}{2}+\frac{\sqrt{17}}{2}\right)$

$\Rightarrow\ |r_1 - r_2| < C_1C_2 < |r_1 + r_2|$

$\Rightarrow$ The two circles intersect each other and the common chord $\overline{AB}$ is the radical axis $S - S' = 0$

$\Rightarrow\ \cancel{x^2} + \cancel{y^2} + 2x + 3y + 1 - \cancel{x^2} - \cancel{y^2} - 4x - 3y - 2 = 0$

$\Rightarrow\ -2x - 1 = 0$

$\Rightarrow\ 2x + 1 = 0$

Let $L = 2x + 1 = 0$

We know that the equation of any circle passing through the points A and B is $S + \lambda(S - S') = 0$ where A and B are the points of intersection of the circles $S = 0$ and $S' = 0$

$\therefore\ S + \lambda(S - S') = 0$ or $S + \lambda L = 0$

$\Rightarrow\ x^2 + y^2 + 2x + 3y + 1 + \lambda(2x + 1) = 0$

$\Rightarrow\ x^2 + y^2 + (2 + 2\lambda)x + 3y + (1 + \lambda) = 0$ ...(1)

If (1) itself is the circle with $\overline{AB}$ as diametre, then its centre $P\left(-\frac{(2+2\lambda)}{2}, \frac{-3}{2}\right)$ lies on the radical axis $L = 0$

$\Rightarrow\ 2\left(-\frac{(2+2\lambda)}{2}\right) + 1 = 0$

$\Rightarrow -2 - 2\lambda + 1 = 0$

$\Rightarrow 2\lambda = -1$

$\Rightarrow \lambda = -\frac{1}{2}$

Substituting λ value in (1), we get the required circle as

$$x^2 + y^2 + \left[2 + 2\left(-\frac{1}{2}\right)\right]x + 3y + \left(1 - \frac{1}{2}\right) = 0$$

$$\Rightarrow\ x^2 + y^2 + x + 3y + \frac{1}{2} = 0$$

$$\Rightarrow\ 2x^2 + 2y^2 + 2x + 6y + 1 = 0.$$

Unit 3

Parabola

Conic Sections

The Circle, parabola, ellipse, hyperbola, a pair of intersecting straight lines; a straight line and a point are called as conic sections because each is a section of a double napped right circular cone with a plane.

Note: A pair of parallel straight lines is not a conic section as there is no plane which cuts the cone along two parallel lines.

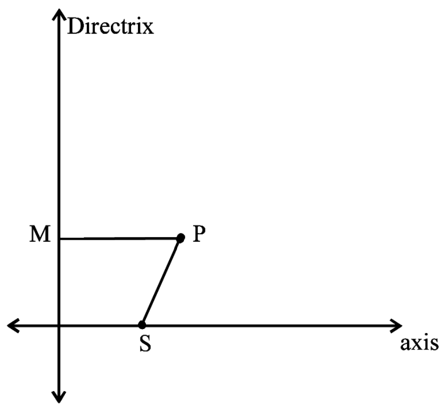
The generated conic sections are a circle, an ellipse, a parabola, a hyperbola. The degenerated conic sections are a point, a straight line, a pair of intersecting straight lines.

Conic

Definition: The locus of a point moving on a plane such that its distances from a fixed point and a fixed straight line in the plane are in a constant ratio 'e' is called a conic.

1. The fixed point is called the focus and is usually denoted by S.
2. The fixed straight line is called the directrix.
3. The constant ratio 'e' is called the eccentricity.
4. The straight line of the plane passing through the focus and perpendicular to the directrix is called the axis.
5. If $e = 1$, the conic is a parabola.

   If $0 < e < 1$, the conic is an ellipse.



If e > 1, the conic is a hyperbola

If e = 0, the conic is a circle.

6. Foci are inside the conic

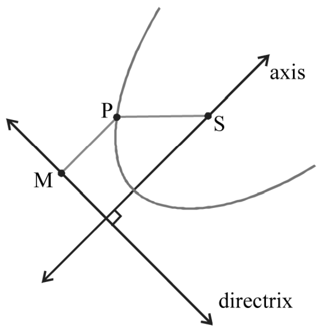
7. Directrices are outside the conic and never intersect the conic.

Parabola

Equation of a parabola in the general form

Let S(α, β) be the focus and $lx+my+n=0$ be the directrix. Then by definition of the parobola,

SP = PM, where P(x, y) is a point on the parabola and PM is the perpendicular distance from P to the directrix.



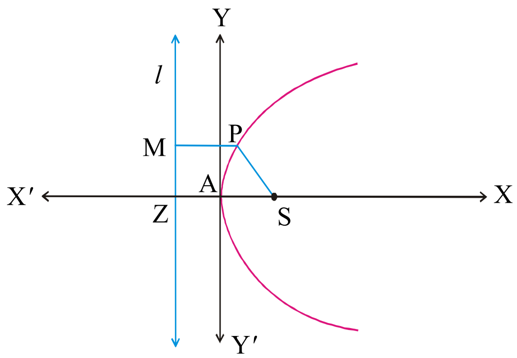
$$\Rightarrow \sqrt{(x-\alpha)^2+(y-\beta)^2} = \frac{|lx+my+n|}{\sqrt{l^2+m^2}}, \text{ when } P=(x,y)$$

$$\Rightarrow (x-\alpha)^2+(y-\beta)^2 = \frac{(lx+my+n)^2}{l^2+m^2}$$ is the equation of parabola which is a second degree equation in x and y. The equation of axis is $m(x-\alpha)-l(y-\beta)=0$.

V Imp LAQ

Theorm. Derive the equation of the Parabola in the standard form as $y^2=4ax$

Proof: Let 'S' be the focus and l be the directix.



Let $\overrightarrow{ZS}$ be the axis which is passing through the focus, S, and perpendicular to the directix l.

Let 'A' be the midpoint of Z, S and 'A' be the origin.

Then ZA = AS.

Let ZA = AS = a and $\overrightarrow{AS}$ be the positive x-axis and $\overrightarrow{AZ}$ be the negative x axis. Let YAY′ be the y-axis,

Then A = (0,0), S = (a,0), Z = (-a,0)

The directrix l is parallel to y-axis and passes through Z. ∴ Its equation is x = –a or x + a = 0.

Let P(x_1, y_1) be any point on the parabola.

Then according to the definition, $\frac{\text{SP}}{\text{PM}} = 1$

$$\Rightarrow \text{SP} = \text{PM}$$

$$\Rightarrow \text{SP}^2 = \text{PM}^2$$

where PM = perpendicular distance from P to the directrix x + a = 0

$$\text{PM} = \frac{|x_1 + a|}{\sqrt{1^2 + 0^2}} = |x_1 + a| \qquad [\because \text{ Formula: } \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}]$$

$$\Rightarrow \text{SP}^2 = \text{PM}^2$$

$$\Rightarrow (x_1 - a)^2 + (y_1 - 0)^2 = |x_1 + a|^2$$

$$\Rightarrow \cancel{x_1^2} + \cancel{a^2} - 2ax_1 + y_1^2 = \cancel{x_1^2} + \cancel{a^2} + 2ax_1$$

$$\Rightarrow y_1^2 = 4ax_1$$

$\therefore$ The locus of P is $y^2 = 4ax$, which is the required standard equation of the parabola.

Nature of the curve of the parabola $y^2 = 4ax$, (a>0)

1. If the curve passes through origin, then $x = 0 \Rightarrow$ y = 0.
2. The y-axis is a tangent to the parabola at the origin.
3. For any positive real value of x, we obtain two values of y of equal magnitude but of opposite in signs. So the curve is symmetric about X-axis and lies in the first and fourth quadrants $(\because x \geq 0)$. The curve doesnot exist on the left side of y-axis.
4. As $x \to \infty, y \to \pm\ \infty$. So the two branches of the parabola lying on opposite sides of the X-axis extend to infinity towards the positive direction of the X-axis. Hence it is an open curve.
5. For the parabola $y^2 = 4ax, (a > 0)$ the focus S is (a, 0), directrix is $x + a = 0$ and axis is $y = 0$. The vertex is A(0, 0).
6. If the vertex is at (h,k) and the axis of the parabola is parallel to X-axis then the equation of the parabola is $(y-k)^2 = 4a(x-h)$.

Definitions:

1. The line joining two points of a parabola is called 'a chord' of a parabola.
2. A chord passing through the focus is called a 'focal chord'.
3. A chord through a point P on the parabola, which is perpendicular to the axis of the parabola, is called the 'double ordinate' of the point P.
4. The double ordinate passing through the focus is called the 'latus rectum' of the parabola.
5. Length of latus rectum is $4a$, $(a > 0)$

   Extremities of latus rectum are $(a, 2a)$ and $(a, -2a)$

Note: When the latus rectum is known, the equation of the parabola is known in its standard form, and the size and shape of the curve are determined accordingly.

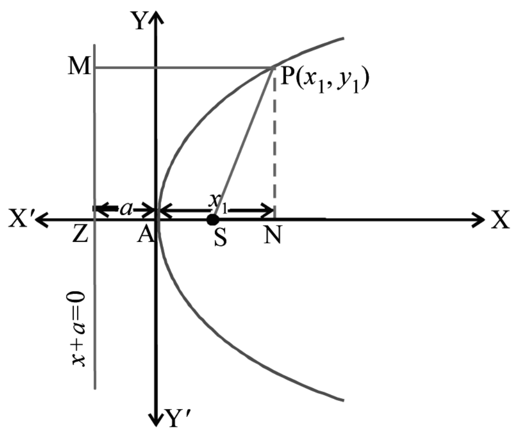
Definition: The distance of a point on the parabola from its focus is called the **'focal distance'** of the point.

Formula: The focal distance of the point $P(x_1, y_1)$ on the parabola $y^2 = 4ax$

whose focus is $S(a, 0)$ is SP

$= PM$

$= x_1 + a$



Parametric equations of the parabola $y^2 = 4ax$

The point $P(at^2, 2at)$ satisfies the equation $y^2 = 4ax$ of a parabola $\forall t \in R$.

$\therefore \boxed{x = at^2, y = 2at}$ are the parametric equations of the parabola $y^2 = 4ax$. Any 'point t' or P(t) is $P(at^2, 2at)$

Notation:

$$S = y^2 - 4ax$$
$$S_1 = yy_1 - 2a(x + x_1)$$
$$S_{11} = y_1^2 - 4ax_1$$
$$S_{12} = y_1y_2 - 2a(x_1 + x_2)$$

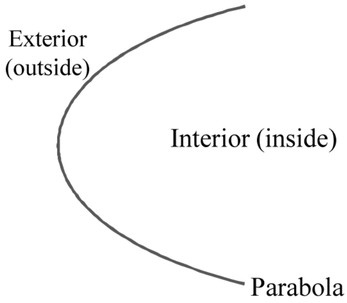
Position of a point w.r.t the parabola $y^2 = 4ax$ or $S = y^2 - 4ax = 0$

The part of the parabola which contains the focus is called the interior of the parabola and the other is called the exterior of the parabola.

(i) $P(x_1, y_1)$ lies outside the parabola $S = y^2 - 4ax = 0 \Leftrightarrow S_{11} > 0$

(ii) The point $P(x_1, y_1)$ lies on the parabola S = 0 $\Leftrightarrow S_{11} = 0$

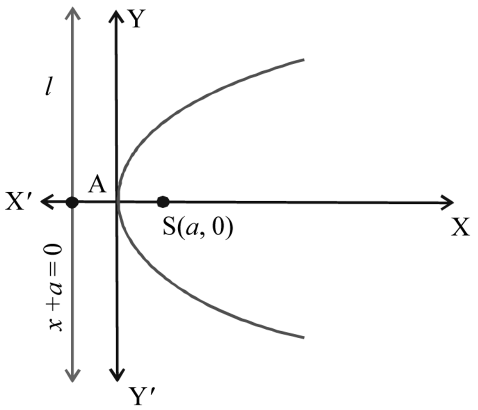
(iii) The point $P(x_1, y_1)$ lies inside the parabola S = 0 $\Leftrightarrow$ $S_{11} < 0$



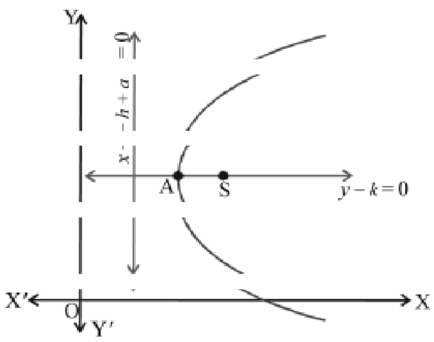
Various forms of the Parabola

(i) The focus is situated on the right side of directrix.

I. The axis is X-axis



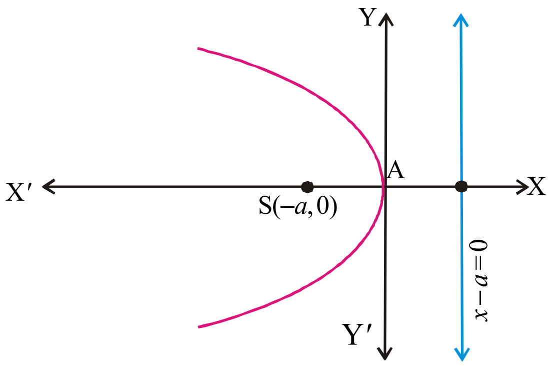
II. The axis is parallel to X axis



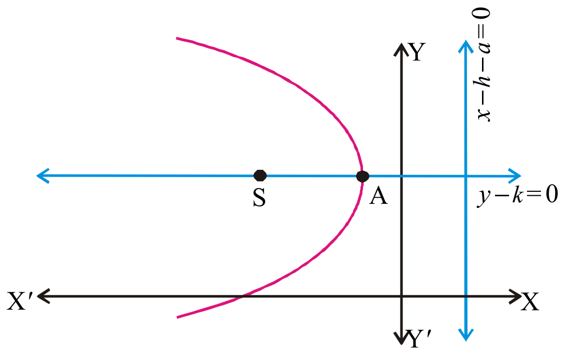
Equation of the Parabola	: $y^2 = 4ax, (a > 0)$	$(y-k)^2 = 4a(x-h), (a > 0)$
Vertex (A)	: (0, 0)	(h, k)
Focus (S)	: $(a, 0)$	$(a+h, k)$
Directrix	: $x = -a$	$x - h = -a$
axis	: $y = 0$	$y - k = 0$
Length of latusrectum	: $4a$	$4a$
Extremities of Latusrectum	: $(a, \pm 2a)$	$(a+h, \pm 2a + k)$

(ii) The focus is situated on the left side of directix

I. The axis is X-axis



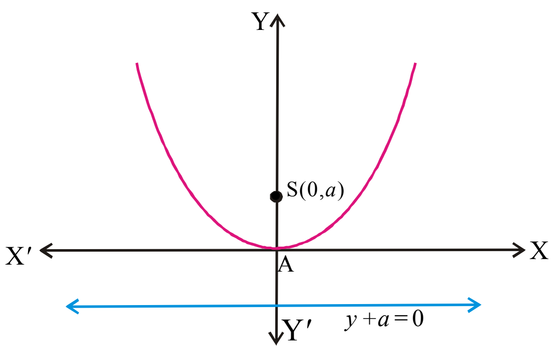
II. The axis is parallel to X axis



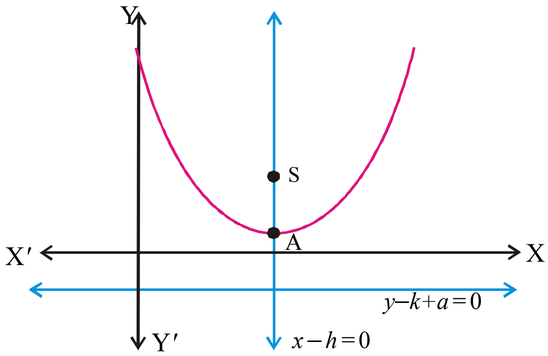
Equation of the Parabola	: $y^2 = -4ax, (a > 0)$	$(y-k)^2 = -4a(x-h), (a > 0)$
Vertex (A)	: (0, 0)	(h, k)
Focus (S)	: $(-a, 0)$	$(-a+h, k)$
Directrix	: $x = a$	$x - h = a$
axis	: $y = 0$	$y - k = 0$
Length of latusrectum	: $4a$	$4a$
Extremities of Latusrectum	: $(-a, \pm 2a)$	$(-a+h, \pm 2a + k)$

(iii) The focus is above the directrix and the axis of the parabola is y-axis or parallel to y-axis

I. The axis is Y-axis

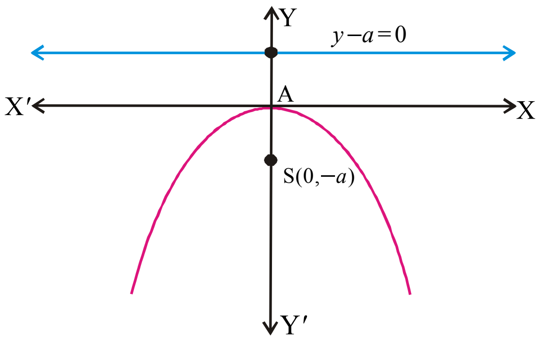


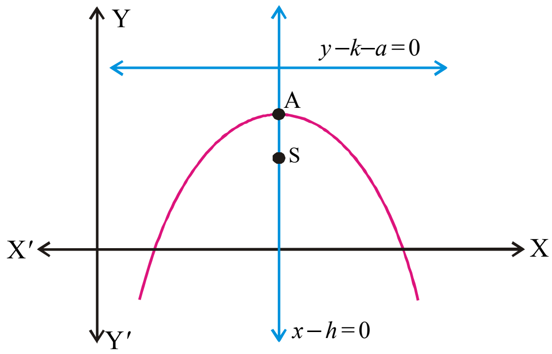
II. The axis is parallel to Y axis



Equation of the Parabola	: $x^2 = 4ay$	$(x-h)^2 = 4a(y-k)$
Vertex (A)	: (0, 0)	(h, k)
Focus (S)	: $(0, a)$	$(h, a+k)$
Directrix	: $y = -a$	$y - k = -a$
axis	: $x = 0$	$x - h = 0$
Length of latusrectum	: $4a$	$4a$
Extremities of Latusrectum	: $(\pm 2a, a)$	$(h \pm 2a, a+k)$

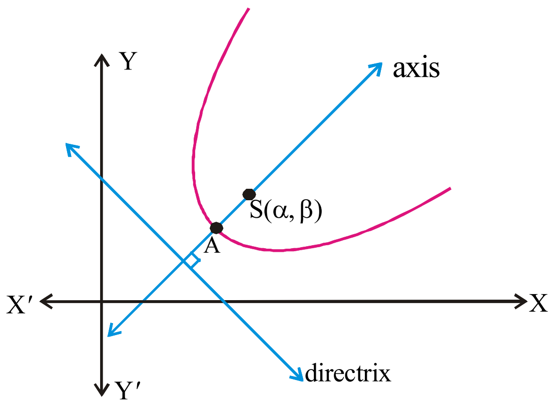
(iv) The focus is below the directrix and the axis of the parabola is y-axis or parallel to y-axis





Equation of the Parabola	: $x^2=-4ay$, $a>0$	$(x-h)^2=-4a(y-k)$, $a>0$
Vertex (A)	: (0, 0)	$(h,\ k)$
Focus (S)	: $(0,-a)$	$(h,-a+k)$
Directrix	: $y=a$	$y-k=a$
axis	: $x=0$	$x-h=0$
Length of latusrectum	: $4a$	$4a$
Extremities of Latusrectum	: $(\pm 2a,-a)$	$(h\pm 2a,-a+k)$

(v) Inclined Parabola



Equation of parabola is

$$(x-\alpha)^2+(y-\beta)^2=\frac{(lx+my+n)^2}{l^2+m^2}$$

Focus (S) = (α,β)

Directix : $lx+my+n=0$

Axis = $m(x-\alpha)-l(y-\beta)=0$

Note: If the focus S lies on the directix, then the locus is a straight line passing through S and perpendicular to the directix. It is a degenerated parabola.

Note: 1) The equation of the parabola whose axis is parallel

(i) to the X-axis is $x=ly^2+my+n$

(ii) to the Y-axis is $y=lx^2+mx+n$ where $l,m,n\in \mathbf{R}$, $l\neq 0$

PROBLEMS

Very Short Answer Questions

I. 1. Find the vertex and focus of $4y^2+12x-20y+67=0$

Sol.: The given parabola is $4y^2+12x-20y+67=0$

$$\Rightarrow 4y^2-20y=-12x-67$$

$$\Rightarrow 4(y^2-5y)=-12x-67$$

$$\Rightarrow 4\left(y^2-2.y\frac{5}{2}+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2\right)=-12x-67$$

$$\Rightarrow 4\left[\left(y-\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2\right]=-12x-67$$

$$\Rightarrow \left[\left(y-\frac{5}{2}\right)^2-\frac{25}{4}\right]=\frac{-12x-67}{4}$$

$$\Rightarrow \left(y-\frac{5}{2}\right)^2=\frac{-12x-67}{4}+\frac{25}{4}$$

$$\Rightarrow \left(y-\frac{5}{2}\right)^2=\frac{-12x-42}{4}$$

$$=\frac{-12\left(x+\frac{42}{12}\right)}{4}$$

$$\Rightarrow \left(y-\frac{5}{2}\right)^2=-3\left(x+\frac{7}{2}\right)$$

This is in the form $(y-k)^2=-4a(x-h)$

where $-k=\frac{-5}{2}, -h=\frac{7}{2}, -4a=-3 \Rightarrow a=\frac{3}{4}$

$\Rightarrow h=\frac{-7}{2}, k=\frac{5}{2}, a=\frac{3}{4}$

For the parabola $(y-k)^2=-4a(x-h)$, the vertex is (h,k) and focus is $(h-a,k)$

$\therefore$ for the given parabola, the vertex is $(h,k)=\left(-\frac{7}{2},\frac{5}{2}\right)$

Focus = S = $(h-a,k)$

$$=\left(\frac{-7}{2}-\frac{3}{4},\frac{5}{2}\right)$$

$$=\left(\frac{-17}{4},\frac{5}{2}\right)$$

2. Find the vertex and focus of $x^2-6x-6y+6=0$

Sol.: Given parabola is $x^2-6x-6y+6=0$

$$\Rightarrow x^2-2.x.3+3^2-3^2-6y+6=0$$

$$\Rightarrow x^2-2.x.3+3^2=6y+3$$

$$\Rightarrow (x-3)^2=6\left(y+\frac{3}{6}\right)$$

$$\Rightarrow (x-3)^2=6\left(y-\left(-\frac{1}{2}\right)\right)$$

This equation is in the form $(x-h)^2=4a(y-k)$

$$\Rightarrow h=3, k=-\frac{1}{2},\ 4a=6$$

$$\Rightarrow a=\frac{3}{2}$$

$\therefore$ The verex is $(h,k)=\left(3,-\frac{1}{2}\right)$

Focus $=(h,k+a)=\left(3,-\frac{1}{2}+\frac{3}{2}\right)=(3,1)$

3. Find the equations of the axis and directix of the parabola $y^2+6y-2x+5=0$.

Sol.: Given parabola is $y^2+6y-2x+5=0$

$$\Rightarrow y^2+2.y.3-2x+5=0$$

$$\Rightarrow y^2+2.y.3+3^2-3^2-2x+5=0$$

$$\Rightarrow (y+3)^2=2x+4$$

$$\Rightarrow (y+3)^2=2(x+2)$$

This equation is in the form $(y-k)^2=4a(x-h)$

$$-k=3, -h=2, 4a=2$$

$$\Rightarrow k=-3, h=-2, a=\frac{1}{2}.$$

The axis of the parabola is $y-k=0 \Rightarrow y+3=0$

The Directrix of the parabola is $x-h=-a$

$$\Rightarrow x+2+\frac{1}{2}=0$$

$$\Rightarrow 2x+5=0$$

Long Answer Questions

4. Find the coordinates of the vertex and focus, the equation of the directrix and axis of the following parabolas.

(i) $y^2+4x+4y-3=0$ (ii) $x^2-2x+4y-3=0$

Sol:. (i) The given parabola $y^2+4x+4y-3=0$

$$\Rightarrow y^2+2.y.2+4x-3=0$$

$$\Rightarrow y^2+2.y.2+2^2-2^2+4x-3=0$$

$$\Rightarrow (y+2)^2-4+4x-3=0$$

$$\Rightarrow (y+2)^2=-4x+7$$

$$\Rightarrow (y+2)^2=-4\left(x+\frac{7}{-4}\right)$$

This equation is in the form $(y-k)^2=-4a(x-h)$

where $-k=2$, $-4a=-4$, $-h=\frac{7}{-4}$

$$\Rightarrow k=-2,\ a=1,\ h=\frac{7}{4}$$

$\therefore$ The vetex is $=(h,k)=\left(\frac{7}{4},-2\right)$

Focus is $=\left(h-a,k\right)=\left(\frac{7}{4}-1,-2\right)$

$$=\left(\frac{3}{4},-2\right)$$

Directrix is $x-h=a \Rightarrow x-\frac{7}{4}=1$

$$\Rightarrow 4x-11=0$$

Axis is $y-k=0 \quad \Rightarrow y+2=0$

Sol.: (ii) The parabola is $x^2 - 2x + 4y - 3 = 0$

$$\Rightarrow x^2 - 2.x.1 + 1^2 - 1^2 + 4y - 3 = 0$$

$$\Rightarrow (x-1)^2 - 1 + 4y - 3 = 0$$

$$\Rightarrow (x-1)^2 = -4y + 4$$

$$\Rightarrow (x-1)^2 = -4(y-1)$$

This equation of parabola is in the form

$$(x-h)^2 = -4a(y-k)$$

$$\Rightarrow h = 1,\ k = 1,\ 4a = 4 \ \Rightarrow \ a = 1$$

$\therefore$ The vertex is $= (h,k) = (1,1)$

Focus is $= (h, k-a) = (1,0)$

Directrix is $y - k = a$

$$\Rightarrow y - k - a = 0 \Rightarrow y - 2 = 0 \text{ Ans}$$

Axis is $x - h = 0$

$$\Rightarrow x - 1 = 0$$

Long Answer Question

5. Find the equation of the parabola whose axis is parallel to X-axis and which passes through the points (–2, 1), (1, 2) and (–1, 3).

Sol. The axis of the prabola is parallel to X-axis

So let the parabola be $ly^2 + my + n = x.$ __(1)

(Since vertex is generally denoted by A, we take the points as P, B, C).

Now it passes through P(–2, 1)

$$\Rightarrow l(1)^2 + m(1) + n = -2$$

$$\Rightarrow l + m + n = -2 \qquad \text{\_\_\_ (2)}$$

Similarly, it passes through B = (1, 2) and C (–1, 3)

$\Rightarrow l(2)^2 + m(2) + n = 1$ and $l(3)^2 + m(3) + n = -1$

$\Rightarrow 4l + 2m + n = 1$ __ (3) and $9l + 3m + n = -1$ __ (4)

Solving (2), (3) and (4) for l, m, n we get

$$\begin{array}{rrrcr} l & + \ m & + \ n & = & -2 \\ 4l & + \ 2m & + \ n & = & 1 \\ - & - & - & & - \\ \hline -3l & - \ m & & = & -3 \end{array} \quad \text{\_\_(5)} \qquad \begin{array}{rrrcr} l & + \ m & + \ n & = & -2 \\ 9l & + \ 3m & + \ n & = & -1 \\ - & - & - & & + \\ \hline -8l & - \ 2m & & = & -1 \end{array} \quad \text{\_\_(6)}$$

$\Rightarrow \quad (3l + m = 3) \times 2 \quad$ __(5)

$8l + 2m = 1 \quad$ __(6)

$$\Rightarrow \begin{array}{r} 6l + 2m = 6 \\ 8l + 2m = 1 \\ - \quad - \quad - \\ \hline -2l \quad = 5 \end{array} \quad \Rightarrow \boxed{l = -\frac{5}{2}}$$

Substituting $l = -\frac{5}{2}$ in (5), we get $\frac{15}{2} - m = -3$

$$\Rightarrow m = \frac{15}{2} + 3 = \frac{21}{2}$$

$$\boxed{m = \frac{21}{2}}$$

From (2), we get $n = -2 - l - m$

$$\Rightarrow n = -2 + \frac{5}{2} - \frac{21}{2} = -10$$

$$\Rightarrow \boxed{n = -10}$$

Substituting the values of l, m, n in (1), we get the required parabola as

$$\frac{-5}{2} y^2 + \frac{21}{2} y - 10 = x$$

$$\Rightarrow \frac{-5y^2 + 21y - 20}{2} = x$$

$$\Rightarrow -5y^2 + 21y - 20 = 2x$$

$$\Rightarrow 5y^2 - 21y + 2x + 20 = 0.$$

6. Find the equation of the parabola whose axis is parallel to y-axis and which passes through the points (4, 5), (–2, 11) and (–4, 21).

Sol. Let the points be P(4, 5), Q(–2, 11) and R(–4, 21).

The axis of the parabola is parallel to y-axis.

So, let the parabola be $lx^2 + mx + n = y$. __(1)

Now it passes through P(4, 5)

$\Rightarrow l(4)^2 + m(4) + n = 5$.

$\Rightarrow 16l + 4m + n = 5 \quad$ __(2)

Again, it passes through Q(–2, 11) and R = (–4, 21)

$\Rightarrow l(-2)^2 + m(-2) + n = 11$ and $l(-4)^2 + m(-4) + n = 21$

$\Rightarrow 4l - 2m + n = 11$ ___(3) $16l - 4m + n = 21$ ___(4)

Solving (2), (3), (4) for $l, m, n,$ we get.

(2) $\Rightarrow 16l + 4m + n = 5$

(3) $\Rightarrow \underline{\underset{-}{4l} \underset{+}{-2m} \underset{-}{+n} = \underset{-}{11}}$

$\underline{12l + 6m = -6}$

$\Rightarrow 6l + 3m = -3$ – (5)

(3) $\Rightarrow 4l - 2m + n = 11$

(4) $\Rightarrow \underline{\underset{-}{16l} \underset{+}{-4m} \underset{-}{+n} = \underset{-}{21}}$

$\underline{-12l + 2m = -10}$

$\Rightarrow -6l + m = -5$ __(6)

(5) $\Rightarrow$ $6l + 3m = -3$

(6) $\Rightarrow$ $\underline{-6l + m = -5}$

$\underline{4m = -8}$ $\Rightarrow \boxed{m = -2}$

Substituting in (6), we get $-6l - 2 = -5$

$\Rightarrow -6l = -5 + 2$

$\Rightarrow -6l = -3$

$\Rightarrow \boxed{l = \frac{1}{2}}$

Substituting the values of l and m in (3),

we get $4\left(\frac{1}{2}\right) - 2(-2) + n = 11$

$\Rightarrow 2 + 4 + n = 11$

$\Rightarrow \boxed{n = 5}$

Substituting the values of *l, m, n* in (1), we get the required parabola as

$$\frac{1}{2}x^2 + (-2)x + 5 = y$$

$$\Rightarrow \frac{x^2 - 4x + 10}{2} = y$$

$$\Rightarrow x^2 - 4x + 10 = 2y$$

$$\Rightarrow x^2 - 4x - 2y + 10 = 0$$

Long Answer Questions

7. Find the equation of the parabola whose focus is (–2, 3) and directrix is the line $2x + 3y - 4 = 0$. Also find the length of the latus rectum and the equation of the axis of the parabola.

Sol: Since focus = $S=(-2,3)$ and directrix is $2x+3y-4=0$ are given, the equation of parabola can be found using the definition : SP = PM.

Where P = (x_1, y_1) is any point on the parabola and PM is the perpendicular distance from P to the directrix.

$\therefore$ SP = PM

$$\Rightarrow \sqrt{(x_1+2)^2+(y_1-3)^2} = \left|\frac{2x_1+3y_1-4}{\sqrt{2^2+3^2}}\right|$$

Squaring on both sides, we get

$$13\left[(x_1+2)^2+(y_1-3)^2\right] = |2x_1+3y_1-4|^2$$

$$\Rightarrow 13(x_1^2+4x_1+4+y_1^2-6y_1+9) = 4x_1^2+9y_1^2+16+12x_1y_1-24y_1-16x_1$$

$$\Rightarrow 9x_1^2-12x_1y_1+4y_1^2+68x_1-54y_1+153=0$$

The locus of P is the equation of required parabola.

$\therefore$ The required parabola is

$$\Rightarrow 9x^2-12xy+4y^2+68x-54y+153=0$$

Length of latus rectum

= $4a$

= $2(2a)$

= 2 x distance from focus to directrix

$$= 2\times\left|\frac{2(-2)+3(3)-4}{\sqrt{2^2+3^2}}\right| \qquad [\because \text{ formula : } \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}} \text{ where } (x_1,y_1)=S\,]$$

$$= 2\times\frac{1}{\sqrt{13}}$$

$$= \frac{2}{\sqrt{13}}$$

To find the equation of axis of the parabola:- We know that the axis is perpendicular to the directrix and passes through the focus.

$\therefore$ The slope of directrix is $\frac{-2}{3} \Rightarrow$ The slope of the axis is $\frac{3}{2}$

$\therefore$ The equation of axis of the parabola with slope $\frac{3}{2}$ and passing through S$(-2,3)$ is

$$y-3=\frac{3}{2}(x+2)$$

$$\Rightarrow 3x-2y+12=0$$

8. Find the equation of the parabola whose focus is S(1, −7) and vertex is A(1, −2).

Sol.: For the parabola, focus = $S=(1,-7)$

Vertex = $A=(1,-2)$

Since the vertex, A and focus, S lie on the axis of the parabola, and since the x-coordinates of S and A are same. $\overleftrightarrow{AS}$ is parallel to y-axis.

So the axis of the parabola is parallel to y-axis.

Since we know the vertex $A=(h,k)=(1,-2)$, the equation of the parabola can be $(x-h)^2=\pm 4a(y-k)$

But the focus = S = (1, −7) always lies inside the parabola.

Since A is above S, the parabola is a downward type of parabola.

So its equation is $(x-h)^2=-4a(y-k)$

Now distance AS = $a=\sqrt{(1-1)^2+(-2+7)^2}=5$

$\therefore$ The equation of the required parabola is

$$(x-1)^2=-4(5)(y+2)$$

$$\Rightarrow (x-1)^2=-20(y+2)$$

Very Short Answer Questions

9. Find the position (interior or exterior or on) of the point(6, −6) with respect to the parabola $y^2=6x$.

Sol. The parabola is $S=y^2-6x=0$

Let $(x_1,y_1)=6,-6)$

$S_{11}=y_1{}^2-6x_1$

$=(-6)^2-6(6)$

$=36-36$

$=0$

$S_{11}=0\Rightarrow$ The point $(6,-6)$ lies on the parabola S = 0.

10. Find the coordinates of the points on the parabola $y^2=8x$ whose focal distance is 10.

Sol. Let $P(x_1,y_1)$ be any point on the parabola $y^2=8x$.

Then $y_1^2=8x_1$ ___(1)

Now comparing $y^2=8x$ with the standard parabola $y^2=4ax$

we get $4a=8\Rightarrow \boxed{a=2}$

The focal distance of P is 10 (given)

$$\Rightarrow x_1 + a = 10$$

$$\Rightarrow x_1 + 2 = 10 \qquad \Rightarrow \boxed{x_1 = 8}$$

Substituting x_1 value in (1) we get $y_1^2 = 8(8) = 64$

$$\Rightarrow y_1 = \pm\sqrt{64} = \pm\ 8.$$

Therefore the points on the parabola whose focal distance is 10 are $(x_1, y_1) = (8,8) \,\&\, (8,-8)$

11. If $\left(\frac{1}{2}, 2\right)$ is one extremity of a focal chord of the parabola $y^2 = 8x$, then find the coordinates of the other extremity.

Sol. Given parabola is $y^2 = 8x$. Comparing it with $y^2 = 4ax$ we get $4a = 8 \Rightarrow \boxed{a = 2}$

Focus = $(a,\ 0)$ = $(2,\ 0)$

Let $\overline{\mathrm{PB}}$ be the focal chord.

Let P = $\left(\mathrm{at}_1^2, 2at_1\right) = \left(\frac{1}{2}, 2\right)$ (parametric coordinates)

$$\Rightarrow at_1^2 = \frac{1}{2},\ \ 2at_1 = 2 \ \ \Rightarrow 2.2.t_1 = 2$$

$$\Rightarrow t_1 = \frac{1}{2}$$

Let $\mathrm{B} = \left(at_2^2, 2at_2\right)$

Since $\overline{\mathrm{PB}}$ is a focal chord, we have $t_1t_2 = -1$

$$\Rightarrow t_2 = \frac{-1}{t_1} = \frac{-1}{1/2} = -2.$$

$$\therefore \mathrm{B} = \left(at_2^2, 2at_2\right) = \left(2(-2)^2, 2(2)(-2)\right)$$

= $(8,-8)$ is the other extremity of the focal chord $\overline{\mathrm{PB}}$.

Long Answer Questions

12. Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}\left|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)\right|$ where $y_1, y_2 y_3$ are the ordinates of its vertices.

Sol. Given Parabola is $y^2 = 4ax$ ___(1)

Let $\mathrm{P}(x_1, y_1)$, $\mathrm{Q}(x_2, y_2)$, $\mathrm{R}(x_3, y_3)$ be three points on the parabola, then $y_1^2 = 4ax_1$, $y_2^2 = 4ax_2$, $y_3^2 = 4ax_3$

$$\Rightarrow x_1 = \frac{y_1^2}{4a}, x_2 = \frac{y_2^2}{4a}, x_3 = \frac{y_3^2}{4a}$$

$$\therefore \text{ Area of } \Delta\text{PQR} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} \frac{y_2^2}{4a} - \frac{y_1^2}{4a} & \frac{y_3^2}{4a} - \frac{y_1^2}{4a} \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$$

$$= \frac{1}{2}\frac{1}{4a}\begin{vmatrix} \left(y_2^2 - y_1^2\right) & \left(y_3^2 - y_1^2\right) \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$$

$$= \frac{1}{8a}\begin{vmatrix} (y_2 - y_1)(y_2 + y_1) & (y_3 - y_1)(y_3 + y_1) \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$$

$$= \frac{1}{8a}(y_2 - y_1)(y_3 - y_1)\begin{vmatrix} y_2 + y_1 & y_3 + y_1 \\ 1 & 1 \end{vmatrix}$$

$$= \frac{1}{8a}\left|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)\right|$$

Second Method:

Let $\text{P} = (x_1, y_1) = (at_1^2, 2at_1)$

$\text{Q} = (x_2, y_2) = (at_2^2, 2at_2)$

$\text{R} = (x_3, y_3) = (at_3^2, 2at_3)$ be 3 points on the parabola $y^2 = 4ax$

Then $x_1 = at_1^2,\ 2at_1 = y_1 \Rightarrow t_1 = \frac{y_1}{2a}$.

$$\Rightarrow x_1 = a\left(\frac{y_1}{2a}\right)^2 = \frac{ay_1^2}{4a^2} = \frac{y_1^2}{4a}$$

$$\text{Area of } \Delta\text{PQR} = \frac{1}{2}\left|\sum x_1(y_2 - y_3)\right|$$

$$= \frac{1}{2}\left|\sum \frac{y_1^2}{4a}(y_2 - y_3)\right|$$

$$= \frac{1}{2}\left|\left(\frac{y_1^2}{4a}(y_2 - y_3) + \frac{y_2^2}{4a}(y_3 - y_1) + \frac{y_3^2}{4a}(y_1 - y_2)\right)\right|$$

$$= \frac{1}{2 \times 4a} \left| y_1^2 (y_2 - y_3) + y_2^2 (y_3 - y_1) + y_3^2 (y_1 - y_2) \right|$$

$$= \frac{1}{8a} \left| y_1^2 y_2 - y_1^2 y_3 + y_2^2 y_3 - y_1 y_2^2 + y_1 y_3^2 - y_2 y_3^2 \right|$$

$$= \frac{1}{8a} \left| (y_1 - y_2)(y_2 - y_3)(y_3 - y_1) \right| \text{Sq.units}$$ because

$(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$

$$= (y_1 - y_2)(y_2 y_3 - y_1 y_2 - y_3^2 + y_1 y_3)$$

$$= \left| y_1^2 y_2 - y_1^2 y_3 + y_2^2 y_3 - y_1 y_2^2 + y_1 y_3^2 - y_2 y_3^2 \right|.$$

$\therefore$ Area of ΔPQR is $\frac{1}{8a} \left| (y_1 - y_2)(y_2 - y_3)(y_3 - y_1) \right|$ Sq. Units.

Hence proved.

Unit **4**

Ellipse

Definition: The conic with eccentricity less than unity is called an ellipse. An ellipse is the locus of a point whose distances from a fixed point and a fixed straight line are in constant ratio 'e' which is less than unity. The fixed point and the fixed straight line are called the focus and the directrix of the ellipse respectively.

Theorem: The equation of the ellipse in the standard formis $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ a>b$

Nature of the curve of the equation of the ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a>b>0)$

(i) The curve intersects X-axis at A(a, 0) and A'(–a, 0), hence AA' = 2a. The curve intersects Y-axis at B(0, b) and B'(0, –b), hence BB'= 2b.

Major and Minor Axes

The line segment AA' and BB' of lengths 2a and 2b respectively are called axes of ellipse.

If a > b, AA' is called major axis and BB' is called minor axis and vice versa if a < b.

Chord, Focal Chord, Latus rectum

1. A line segment joining two points on the ellipse is called a 'chord' of the ellipse.
2. A chord passing through one of the foci is called a 'focal chord".
3. A focal chord perpendicular to the major axis of the ellipse is called a**latus rectum**. An ellipse has two **latus recta.**

Note: The foci S, S', the vertices A, A' lie on the major axis of the ellipse.

The standard equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a>b)$**.**

It is horizontal ellipse $b^2 = a^2(1-e^2)$, $0 < e < 1$

C = Centre = (0, 0), S = focus = (ae, 0), S' = focus = (–ae, 0)

Distance between the foci = Distance SS' = 2ae.

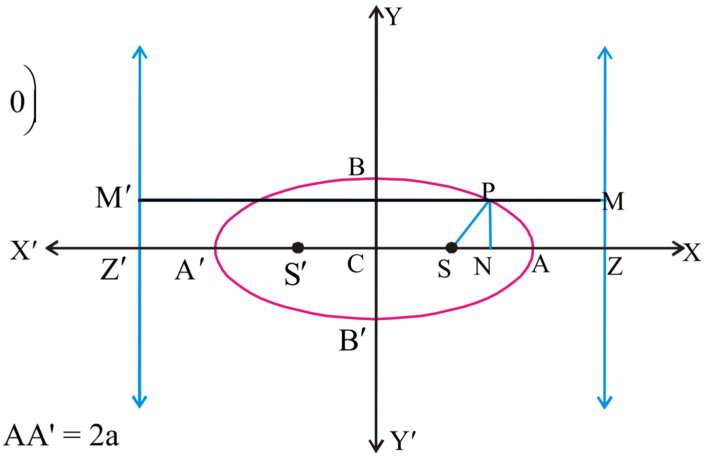
Directrix :

$\Rightarrow CZ = \frac{a}{e}$, $Z = \left(\frac{a}{e}, 0\right)$ $\Rightarrow Z' = \left(-\frac{a}{e}, 0\right)$

Directrices $x = \frac{a}{e}$ and $x = -\frac{a}{e}$

So, distance between the directrices

= distance ZZ' = $2\frac{a}{e}$.



Length of major axis : Distance AA' = 2a

[A = (a, 0), A' = (–a, 0)]

Length of minor axis : Distance BB' = 2b [$\because$ B = (0, b), B' = (0, –b)]

Centre of the ellipse = C = midpoint of S, S'

= midpoint of A, A'

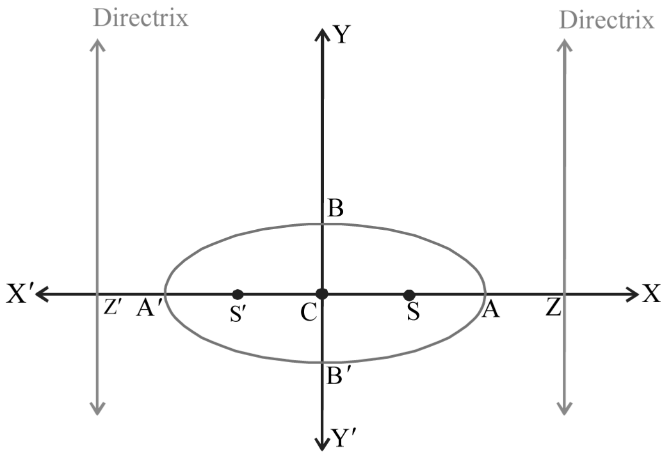
= midpoint of Z, Z'

Various forms of the ellipse

If $a = b$, then the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a circle ($x^2 + y^2 = a^2$) with centre at origin and having radius 'a' and we are familiar with circles. We assumed $a \neq b$ and in the following discussion, we describe different forms of the ellipse.

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b > 0)$

Major axis	along X-axis
Length of major axis (AA′)	$2a$
Minor axis	along Y-axis
Length of minor axis (BB′)	$2b$
Centre	C = (0, 0)
Foci	S = (ae, 0), S′ = ($-ae$, 0)
Equation of the directrices	$x = a/e$ $x = -a/e$
Eccentricity	$e = \sqrt{\frac{a^2 - b^2}{a^2}}$



Fig

(ii) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\,(0<a<b)$

Major axis	along Y-axis
Length of major axis (BB')	$2b$
Minor axis	along X-axis
Length of minor axis (AA')	$2a$
Centre	$C=(0,0)$
Foci	$S=(0,be)$ $S'=(0,-be)$
Equation of the directrices	$y=b/e$ $y=-b/e$
Eccentricity	$e=\sqrt{\dfrac{b^2-a^2}{b^2}}$

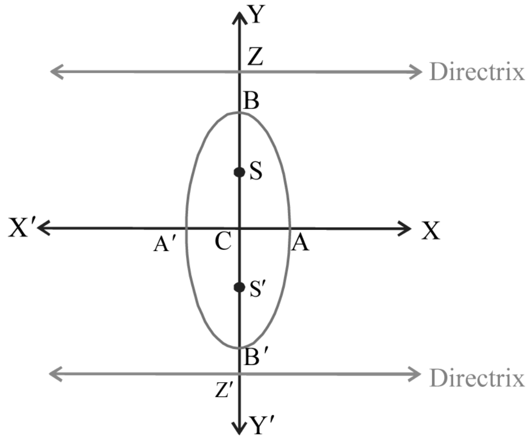


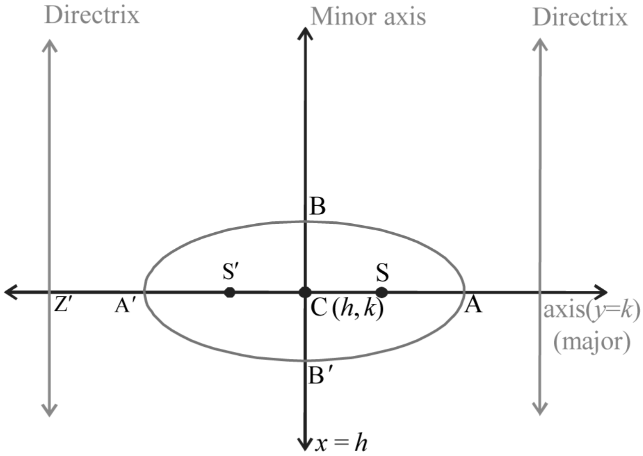
Fig.

Centre not at the origin

If the centre is at (h,k) and the axes of the ellipse are parallel to the X-and Y - axis, then by shifting the origin to (h,k) by translation of axes and using the results (i) and (ii) above, the following results (iii) and (iv) can be obtained.

(iii) $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1,\ (a>b>0)$

Major axis	along $y=k$
Length of major axis (AA')	$2a$
Minor axis	along $x=h$
Length of minor axis (BB')	$2b$
Centre	$C=(h,k)$
Foci	$S=(h+ae,k)$ $S'=(h-ae,k)$
Equation of the directrices	$x=h+a/e$ $x=h-a/e$
Eccentricity	$e=\sqrt{\dfrac{a^2-b^2}{a^2}}$



Fig

(iv) $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1,\ (0<a<b)$

Major axis	along $x=h$
Length of the major axis (BB′)	$2b$
Minor axis	along $y=k$
Length of the minor axis (AA′)	$2a$
Centre	$C=(h,k)$
Foci	$S=(h,k+be)$ $S'=(h,k-be)$
Equation of the directrices	$y=k+b/e$ $y=k-b/e$
Eccentricity	$e=\sqrt{\dfrac{b^2-a^2}{b^2}}$

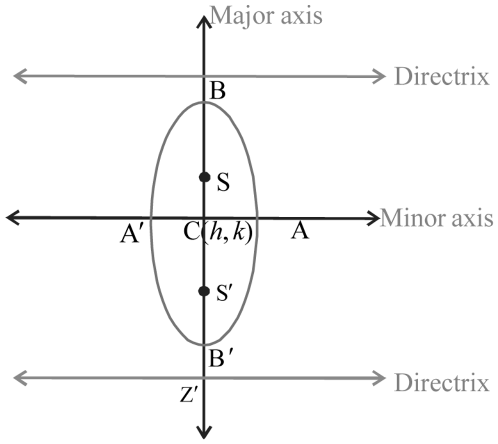
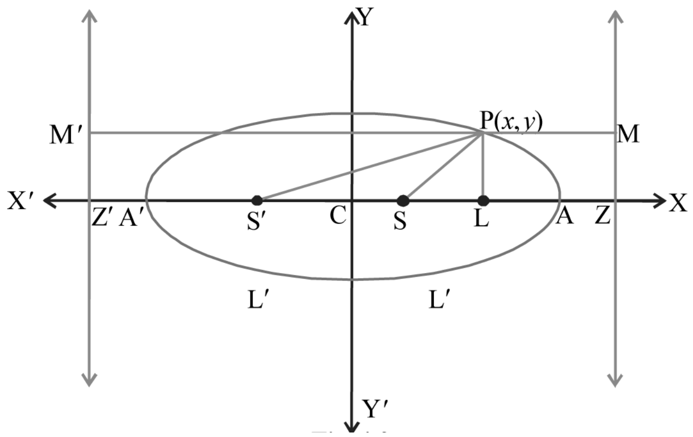


Fig.

Theorem : If P(x, y) is any point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\ (a>b)$ whose foci are S and S', then prove that SP + S'P is a constant.

Proof:



Let S, S' be the foci and ZM, Z'M' be the corresponding directrices of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,\ (a>b)$.

Join SP and S'P where P(x, y) is a point on the ellipse. Draw PL perpendicular to x-axis and M'MP perpendicular to the two directrices.

By definition of the ellipse,

$$SP = e(PM) = e(LZ) = e(CZ - CL) = e\left(\frac{a}{e} - x\right) = a - xe$$

$$S'P = e(PM') = e(LZ') = e(CL + CZ') = e\left(x + \frac{a}{e}\right) = a + xe$$

$\therefore$ SP + S'P = $a - xe + a + xe = 2a$ = constant = length of major axis

(or) SP + S'P = e(PM + PM') = e(MM')

$$= e \times \text{distance between the directrices} = e \times \frac{2a}{e} = 2a = \text{constant}$$

Auxiliary circle

The circle described on the major axis of an ellipse as diameter is called 'Auxiliary Circle' of the ellipse.

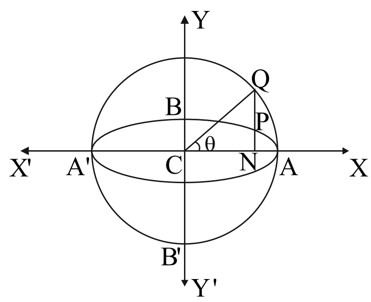
The equation of the Auxiliary Circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ (a > b)$ is $x^2 + y^2 = a^2$.

Eccentric angle and Parametric equation

Let P be any point on the ellipse. Draw PN perpendicular to the major axis and produce it to meet the auxiliary circle at Q. Then angle ACQ is called the eccentric angle of the point P. $0 \le \theta < 2\pi$.

x = a cosθ, y = b sinθ are known as the parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where θ is called the parameter.

Any point P on the ellipse is (acosθ, bsin θ) = point θ = P(θ)



Notation

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

$$S_1 = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$$

$$S_{11} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

$$S_{12} = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1$$

The point P(x_1, y_1) lies outside, on or inside the ellipse S = 0 according as S_{11} is positive, zero or negative respectively.

Problems

1. Find the equation of the ellipse with focus at $(1,-1), e=\frac{2}{3}$ and directrix as $x+y+2=0$.

Sol. Let the focus S = $(1,-1)$, $e=\frac{2}{3}$ and directrix is $x+y+2=0$.

Let $P(x_1, y_1)$ be any point on the ellipse.

Then according to the definition, $\frac{SP}{PM}=e$ .... (1)

where PM is the perpendicular distance from P to the directrix.

$$\therefore PM=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

$$=\frac{|x_1+y_1+2|}{\sqrt{1^2+1^2}}$$

From (1)

$\therefore$ SP = e PM

$$\Rightarrow \sqrt{(x_1-1)^2+(y_1+1)^2}=\frac{2}{3}\frac{|x_1+y_1+2|}{\sqrt{2}}$$

Squaring on both sides, we get

$$\Rightarrow (x_1-1)^2+(y_1+1)^2=\frac{4|x_1+y_1+2|^2}{9\times 2}$$

$$\Rightarrow 7x_1^{\,2}+7y_1^{\,2}-4x_1y_1-26x_1+10y_1+10=0$$

$\therefore$ The locus of $P(x_1, y_1)$ is

$7x^2+7y^2-4xy-26x+10y+10=0$

which is the required equation of the ellipse.

2. Find the equation of the ellipse in the standard form whose distance between foci is 2 and the length of latus rectum is $\frac{15}{2}$.

Sol. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Distance between foci = 2ae = 2 $\Rightarrow$ ae = 1.

Length of the latus rectum $=\frac{2b^2}{a}=\frac{15}{2}$

$\Rightarrow 4b^2=15a$ $\qquad [\because b^2=a^2(1-e^2)=a^2-a^2e^2]$

$$\Rightarrow 4\left[a^2 - a^2e^2\right] = 15a$$

$$\Rightarrow 4a^2 - 4a^2e^2 - 15a = 0$$

$$\Rightarrow 4a^2 - 15a - 4 = 0 \qquad [\because a^2e^2 = (ae)^2 = 1^2 = 1]$$

$$\Rightarrow (4a+1)(a-4) = 0$$

$$\Rightarrow a = \frac{-1}{4} \text{ or } 4$$

$$\Rightarrow a = 4 \qquad [\because a \text{ is} + \text{ve}, a \neq \frac{-1}{4}]$$

$$\Rightarrow b^2 = a^2 - a^2e^2 = 16 - 1 = 15$$

$\therefore$ The required ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{15} = 1$$

3. Find the equation of the ellipse in the standard form such that the distance between foci is 8 and distance between directrices is 32.

Sol. Let the ellipse in the standard form be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Distance between the foci = 2ae = 8 $\Rightarrow$ ae = 4.

Distance between the directrices $\frac{2a}{e} = 32 \Rightarrow \frac{a}{e} = 16$.

Now, $ae \times \frac{a}{e} = 4 \times 16 \Rightarrow a^2 = 64 \Rightarrow a = 8$.

$b^2 = a^2(1-e^2) = a^2 - a^2e^2 = 64 - (4)^2 = 64 - 16 = 48$.

$\therefore$ The ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{64} + \frac{y^2}{48} = 1$.

4. Find the eccentricity of the ellipse (in standard form) if its length of latus rectum is equal to half of its major axis.

Sol. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Given, length of latus rectum = $\frac{1}{2}$ × length of major axis

$$\Rightarrow \frac{2b^2}{a} = \frac{1}{2}(2a) \Rightarrow 2b^2 = a^2$$

$$\Rightarrow 2\left[a^2(1-e^2)\right] = a^2$$

$$\Rightarrow 2(1-e^2) = 1$$

$$\Rightarrow 1-e^2 = \frac{1}{2}$$

$$\Rightarrow e^2 = 1-\frac{1}{2} = \frac{1}{2} \Rightarrow e = \frac{1}{\sqrt{2}}$$

$\therefore$ The eccentricity of the ellipse is, $e = \frac{1}{\sqrt{2}}$.

5. Find the equation of the ellipse in the standard form, if it passes through the points (–2, 2) and (3, –1).

Sol. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the standard form.

It passes through the points (–2, 2) and (3, –1).

$$\Rightarrow \frac{(-2)^2}{a^2} + \frac{2^2}{b^2} = 1 \text{ and } \frac{(3)^2}{a^2} + \frac{(-1)^2}{b^2} = 1$$

$$\Rightarrow \left(4\times\frac{1}{a^2}\right) + \left(4\times\frac{1}{b^2}\right) = 1 \text{ and } \left(9\times\frac{1}{a^2}\right) + \left(1\times\frac{1}{b^2}\right) = 1$$

Let $\frac{1}{a^2} = m,\ \frac{1}{b^2} = n$

$$\begin{array}{ll} \text{Then} & 4m + 4n = 1 \\ \text{and} & \underline{9m + \ \ n = 1} \\ & 4m + 4n = 1 \\ & 36m + 4n = 4 \\ & \underline{-\quad\ -\quad\ -} \\ & -32m = -3 \end{array}$$

$$\Rightarrow m = \frac{3}{32} \Rightarrow n = 1-9m = 1-9\times\frac{3}{32} = \frac{5}{32}$$

$\therefore$ The required ellipse is $x^2\left(\frac{1}{a^2}\right) + y^2\left(\frac{1}{b^2}\right) = 1$

$$\Rightarrow x^2 m + y^2 n = 1$$

$$\Rightarrow x^2\left(\frac{3}{32}\right) + y^2\left(\frac{5}{32}\right) = 1$$

$\Rightarrow 3x^2 + 5y^2 = 32$, is the required ellipse.

6. If the length of the major axis of an ellipse is 3 times the length of its minor axis, then find the eccentricity of the ellipse.

Sol. Let the ellipse in the standard form be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Given length of major axis = 3 × length of minor axis

$$\Rightarrow 2a = 3 \times 2b \Rightarrow a = 3b$$

But $$b^2 = a^2(1-e^2)$$

$$\Rightarrow b^2 = (3b)^2(1-e^2)$$

$$\Rightarrow b^2 = 9b^2(1-e^2)$$

$$\Rightarrow \frac{b^2}{9b^2} = (1-e^2)$$

$$\Rightarrow e^2 = 1-\frac{1}{9}=\frac{8}{9}$$

$$\Rightarrow e = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} = \text{eccentricity of the ellipse.}$$

7. Find the length of the major axis, minor axis, latus rectum, eccentricity, coordinates of centre, foci and the equations of directrices of the following ellipse.

(i) $9x^2 + 16y^2 = 144$ (ii) $4x^2 + y^2 - 8x + 2y + 1 = 0$ (iii) $x^2 + 2y^2 - 4x + 12y + 14 = 0$

Sol. **(i)** Given ellipse is $9x^2 + 16y^2 = 144$

$$\Rightarrow \frac{9x^2}{144}+\frac{16y^2}{144}=1 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1$$

Comparing this equation with the standard equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

we get $a^2 = 16$, $b^2 = 9 \Rightarrow a = 4$, $b = 3$.

$a > b \Rightarrow b^2 = a^2(1-e^2)$

$$\Rightarrow 9 = 16(1-e^2) \qquad \Rightarrow \frac{9}{16} = 1-e^2$$

$$\Rightarrow e^2 = 1-\frac{9}{16}=\frac{7}{16}$$

$$\Rightarrow e = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

$a > b \Rightarrow$ The ellipse is a horizontal ellipse.

(i) Length of major axis = $2a = 8$.

(ii) Length of minor axis = $2b = 6$.

(iii) Length of latus rectum $= \dfrac{2b^2}{a} = \dfrac{2(9)}{4} = \dfrac{9}{2}$

(iv) Eccentricity $= e = \dfrac{\sqrt{7}}{4}$

(v) Centre = (0, 0)

(vi) Foci = (±ae, 0) = $(\pm\sqrt{7}\,, 0)$

(vii) Directrices : $x = \pm\dfrac{a}{e} \Rightarrow x = \pm\dfrac{16}{\sqrt{7}} \Rightarrow \sqrt{7}x = \pm 16$

(ii) Given ellipse is $4x^2 + y^2 - 8x + 2y + 1 = 0$

Writing it in the standard form:

$(4x^2 - 8x) + (y^2 + 2y) + 1 = 0$

$\Rightarrow 4(x^2 - 2x) + (y^2 + 2.y.1 + 1^2 - 1^2) + 1 = 0$

$\Rightarrow 4(x^2 - 2.x.1 + 1^2 - 1^2) + (y+1)^2 = 0$

$\Rightarrow 4[(x-1)^2 - 1] + (y+1)^2 = 0$

$\Rightarrow 4(x-1)^2 - 4 + (y+1)^2 = 0$

$\Rightarrow 4(x-1)^2 + (y+1)^2 = 4$

$$\Rightarrow \frac{4(x-1)^2}{4} + \frac{(y+1)^2}{4} = \frac{4}{4}$$

$$\Rightarrow \frac{(x-1)^2}{1} + \frac{(y+1)^2}{4} = 1$$

Comparing with the standard equation $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

We get $h = 1, -k = 1 \Rightarrow k = -1,$

$a^2 = 1 \Rightarrow a = 1,\; b^2 = 4 \Rightarrow b = 2$

$\Rightarrow a < b.$

$\Rightarrow$ The ellipse is a vertical ellipse.

$a^2 = b^2(1-e^2) \Rightarrow 1 = 4(1-e^2)$

$$\Rightarrow \frac{1}{4} = 1 - e^2 \Rightarrow e^2 = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow e = \frac{\sqrt{3}}{2}$$

(i) Length of major axis $= 2b = 4$.

(ii) Length of minor axis $= 2a = 2$.

(iii) Length of latus rectum $= \dfrac{2a^2}{b} = \dfrac{2(1)}{2} = 1$

(iv) Eccentricity $= e = \dfrac{\sqrt{3}}{2}$

(v) Centre = (h, k) = (1, –1)

(vi) Foci = (h, k±be) = $(1,\ -1 \pm \sqrt{3}\ )$

(vii) Directrices : $y - k = \pm\dfrac{b}{e} \Rightarrow y + 1 = \pm\dfrac{4}{\sqrt{3}} \Rightarrow \sqrt{3}y + \sqrt{3} = \pm 4$

(iii) $x^2 + 2y^2 - 4x + 12y + 14 = 0$

Writing in the standard from, we get

$x^2 - 4x + 2y^2 + 12y + 14 = 0$

$\Rightarrow (x^2 - 2.x.2) + 2(y^2 + 6y) + 14 = 0$

$\Rightarrow (x^2 - 2.x.2 + 2^2 - 2^2) + 2(y^2 + 2.y.3 + 3^2 - 3^2) + 14 = 0$

$\Rightarrow (x-2)^2 - 4 + 2[(y+3)^2 - 9] + 14 = 0$

$\Rightarrow (x-2)^2 - 4 + 2(y+3)^2 - 18 + 14 = 0$

$\Rightarrow (x-2)^2 + 2(y+3)^2 = 8$

$\Rightarrow \dfrac{(x-2)^2}{8} + \dfrac{2(y+3)^2}{8} = 1$

$\Rightarrow \dfrac{(x-2)^2}{8} + \dfrac{(y+3)^2}{4} = 1$

Comparing with the standard equation $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

We get $h = 2, k = -3$

$a^2 = 8 \Rightarrow a = 2\sqrt{2}$, $b^2 = 4 \Rightarrow b = 2$

$\Rightarrow a > b \Rightarrow$ The ellipse is a horizontal ellipse.

$b^2 = a^2(1 - e^2) \Rightarrow 4 = 8(1 - e^2)$

$\Rightarrow \dfrac{4}{8} = 1 - e^2 \Rightarrow e^2 = 1 - \dfrac{4}{8} = \dfrac{4}{8} = \dfrac{1}{2} \Rightarrow e = \dfrac{1}{\sqrt{2}}$

(i) Length of major axis = $2a = 4\sqrt{2}$.

(ii) Length of minor axis = $2b = 4$.

(iii) Length of latus rectum $= \dfrac{2b^2}{a} = \dfrac{2.4}{2\sqrt{2}} = \dfrac{4}{\sqrt{2}} = \dfrac{4}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{4\sqrt{2}}{2} = 2\sqrt{2}$

(iv) Eccentricity $= e = \dfrac{1}{\sqrt{2}}$

(v) Coordinates of Centre = C = (h, k) = (2, –3)

(vi) Coordinates of Foci = (h±ae, k) = (2±2, –3) = (4, –3), (0, –3)

(vii) Equation of Directrices : $x-h=\pm\frac{a}{e}\Rightarrow x-2=\pm\frac{2\sqrt{2}}{1/\sqrt{2}}\Rightarrow x-2=\pm4$

$\Rightarrow x-2=+4,\ x-2=-4$

$\Rightarrow x-6=0,\quad x+2=0$ are the directrices.

8. Find the equation of the ellipse in the form of $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ given the following data

(i) Centre = (2, –1); one end of major axis = (2, –5) , $e=\frac{1}{3}$

(ii) Centre = (4, –1); one end of major axis = (–1, –1) , passing through (8, 0)

(iii) Centre = (0, –3); $e=\frac{2}{3}$, semi minor axis = 5

(iv) Centre = (2, –1); $e=\frac{1}{2}$, Length of latus rectum = 4

Sol. **(i)** Centre, C = (2, –1) = (h, k)

one end of major axis = B = (2, –5) = vertex

Since the x-coordinate of C and B are same,

the line $\overleftrightarrow{CB}$ is parallel to y-axis.

We know that, C and B lie on major axis.

$\therefore$ Major axis is parallel to y-axis.

The ellipse is a vertical ellipse.

$\text{CB}=b=\sqrt{(2-2)^2+(-1+5)^2}=4$

Given $e=\frac{1}{3}$

$\therefore a^2=b^2(1-e^2)$

$\Rightarrow a^2=16\left(1-\frac{1}{9}\right)=16\times\frac{8}{9}=\frac{128}{9}$

$\therefore$ The ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1\Rightarrow\frac{(x-2)^2}{\frac{128}{9}}+\frac{(y+1)^2}{16}=1$

$\Rightarrow\frac{9(x-2)^2}{128}+\frac{(y+1)^2}{16}=1$

(ii) Centre of the ellipse = C = (h, k) = (4, –1)

one end of major axis = A = (–1, –1)

Since the y-coordinate of C and A are same,

the line $\overleftrightarrow{CA}$ is parallel to x-axis.

We know that, C and A lie on major axis.

$\therefore$ Major axis is parallel to x-axis.

Distance $CA = a = \sqrt{(4+1)^2+(-1+1)^2} = 5$

$\therefore$ The ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \Rightarrow \frac{(x-4)^2}{25}+\frac{(y+1)^2}{b^2}=1$ .... (1)

It passes through (8, 0)

$$\Rightarrow \frac{(8-4)^2}{25}+\frac{1^2}{b^2}=1 \Rightarrow \frac{1^2}{b^2}=1-\frac{16}{25}=\frac{9}{25}$$

Substituting $\frac{1}{b^2}=\frac{9}{25}$ in (1) we get the required ellipse as

$$\Rightarrow \frac{(x-4)^2}{25}+(y+1)^2\times\frac{1}{b^2}=1 \Rightarrow \frac{(x-4)^2}{25}+(y+1)^2\times\frac{9}{25}=1$$

$$\Rightarrow (x-4)^2+9(y+1)^2=25$$

(iii) Centre = (0, –3); $e=\frac{2}{3}$, semi minor axis = 5

<u>Case (i)</u>: When $a > b$

Centre = C = (h, k) = (0, –3)

Length of semi minior axis $=\frac{2b}{2}=b=5$ (when the ellipse is a horizontal ellipse)

$$e=\frac{2}{3} \Rightarrow b^2=a^2(1-e^2)$$

$$\Rightarrow 25=a^2\left(1-\frac{4}{9}\right) \Rightarrow 25=a^2\left(\frac{5}{9}\right) \Rightarrow a^2=25\times\frac{9}{5}=45$$

$\therefore$ The required ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \Rightarrow \frac{(x-0)^2}{45}+\frac{(y+3)^2}{25}=1$.

<u>Case (ii):</u> When $a < b$ Centre = C = (h, k) = (0, –3)

Length of semi minior axis $=a=5$ (when the ellipse is a vertical ellipse)

$$e=\frac{2}{3} \Rightarrow a^2=b^2(1-e^2)$$

$$\Rightarrow 25=b^2\left(1-\frac{4}{9}\right) \Rightarrow b^2=45$$

$\therefore$ The required ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \Rightarrow \frac{(x-0)^2}{25}+\frac{(y+3)^2}{45}=1$.

(iv) Centre = (2, –1); $e=\frac{1}{2}$, Length of latus rectum = 4

Case (i): When $a > b$ Centre = C = (h, k) = (2, –1), $e=\frac{1}{2}$

Length of latus rectum $=\frac{2b^2}{a}=4$ (for horizontal ellipse)

$\Rightarrow 2b^2 = 4a \Rightarrow b^2 = 2a$

$\Rightarrow a^2(1-e^2) = 2a \Rightarrow a\left(1-\frac{1}{4}\right)=2$

$\Rightarrow \frac{3a}{4}=2 \Rightarrow a=\frac{8}{3}$

$\therefore b^2 = 2a = \frac{16}{3}$

$\therefore$ The required ellipse is $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \Rightarrow \frac{(x-2)^2}{\frac{64}{9}}+\frac{(y+1)^2}{\frac{16}{3}}=1$.

$$\Rightarrow \frac{9(x-2)^2}{64}+\frac{3(y+1)^2}{16}=1$$

Case (ii): When $a < b$ Centre = C = (h, k) = (2, –1), $e=\frac{1}{2}$

Length of latus rectum $=\frac{2a^2}{b}=4$ (for vertical ellipse)

$\Rightarrow 2a^2 = 4b \Rightarrow a^2 = 2b$

$\Rightarrow b^2(1-e^2) = 2b \Rightarrow b\left(1-\frac{1}{4}\right)=2 \Rightarrow b=\frac{8}{3}$

$\therefore a^2 = 2b = \frac{16}{3}$

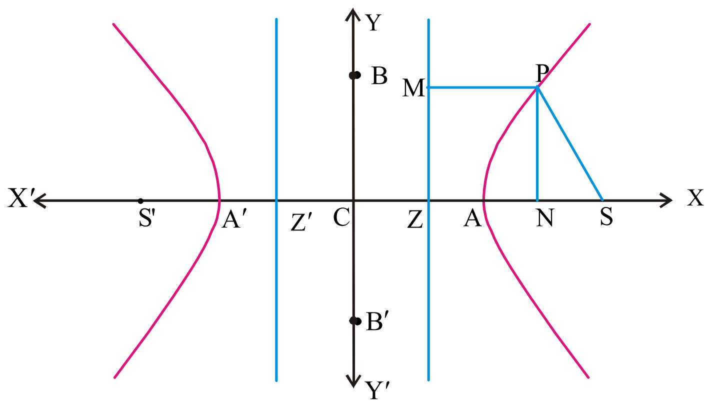
$\therefore$ The required ellipse is $\frac{(x-2)^2}{\frac{16}{3}}+\frac{(y+1)^2}{\frac{64}{9}}=1 \Rightarrow \frac{3(x-2)^2}{16}+\frac{9(y+1)^2}{64}=1$.

Unit 5

Hyperbola

- Hyperbola is a conic in which the eccentricity is greater than unity.
- Hyperbola is the locus of a point that moves so that the ratio of the distance from a fixed point to its distance from a fixed straight line is greater than 1.
- The fixed point is called focus, the fixed straight line is called directrix.
- The equation of hyperbola in the standard form is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{where} \quad b^2=a^2\left(e^2-1\right) \text{ and } e>1$$



Trace of the Curve:

The hyperbola in the standard form is $S=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ where a > 0, b > 0 and $b^2=a^2(e^2-1)$

(i) The hyperbola cuts the x-axis at A(a, 0) and A^1($-a$, 0) called as vertices.

(ii) $x=0 \Rightarrow y=\pm\sqrt{-b^2} \Rightarrow$ The curve does not intersect y-axis.

(iii) $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$ then y is real $\Leftrightarrow x^2 - a^2 \geq 0$ $\Leftrightarrow x \leq -a$ or $x \geq a$

The curve does not exist between the vertical lines $x = -a$ and $x = a$. Further from

$x = \pm\frac{a}{b}\sqrt{y^2 + b^2} \Rightarrow y \to \pm\infty$ when $x \to \pm\infty$ y is real $\Rightarrow x$ is real $\Rightarrow$ each horizontal line $y = k$ intersects the hyperbola at two points. Also $x \to \pm\infty \Rightarrow y \to \pm\infty$ i.e. the curve is unbounded.

(iv) The curve is symmetric about X-axis and also about Y-axis. The curve consists of two symmetrical branches each extending to infinity in two directions.

(v) $\overline{AA^1}$ is called as Transverse axis of the hyperbola

$\overline{BB^1}$ is called as conjugate axis where BC = $B^1C = b = a\sqrt{e^2 - 1}$ and B, B' lie on Y-axis.

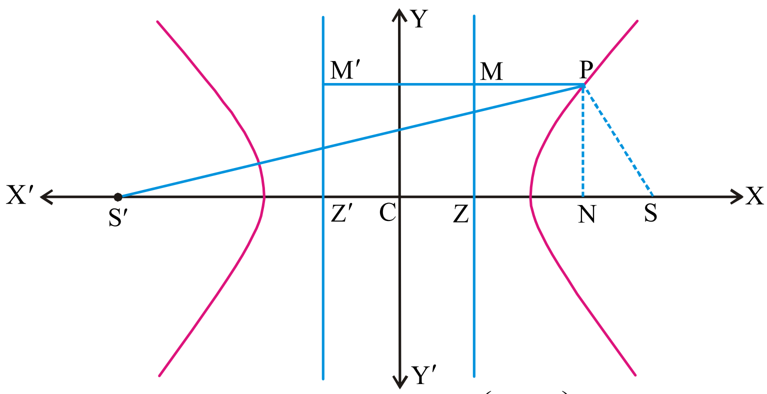
(vi) As in the ellipse, the symmetry of the curve about its axis shows that it has two foci, $S = (ae, 0)$, $S' = (-ae, 0)$ and two directrices $x = \pm\frac{a}{e}$.

(vii) C is called the centre of the hyperbola. It is the point of intersection of the transverse and conjugate axis. C bisects every chord of the hyperbola that passes through itself.

Theorem:

Prove that the difference of the focal distances of any point on the hyperbola is constant.

Proof: Let P(x, y) be any point on the hyperbola whose centre is the origin C, foci are S, S', directrices are $\overleftrightarrow{ZM}$ and $\overleftrightarrow{Z'M'}$. Let PN, PM, PM' be the perpendiculars drawn from P upon x-axis and the two directices respectively.



Now SP = e(PM) = e(NZ) = e(CN–CZ) = $e\left(x - \frac{a}{e}\right) = ex - a$.

$S'P$ = e(PM′) = e(NZ′) = e(CN + CZ′) = $e\left(x + \frac{a}{e}\right) = ex + a$.

$\therefore S'P - SP = (ex + a) - (ex - a) = 2a = \text{constant}$.

$\therefore$ The difference of the focal distances of the point P is a constant.

Note: Hyperbola is also defined as the Locus of a point, the difference of whose distances from two fixed points is constant.

Notation

$$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$$

$$S_1 = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$$

$$S_{11} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

$$S_{12} = \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} - 1$$

Very Short Answer Questions

Definition Rectangular Hyperbola

1. Define Rectangular Hyperbola and find its eccerticity.

Ans. In a hyperbola, if the length of the transverse axis (2a) is equal to the length of the conjugate axis (2b), then the hyperbola is called as Rectangular Hyperbola.

Its equation is $x^2 - y^2 = a^2$

$$b = a \Rightarrow a^2 = a^2(e^2 - 1) \Rightarrow e^2 - 1 = 1 \Rightarrow e^2 = 2 \Rightarrow e = \sqrt{2}$$

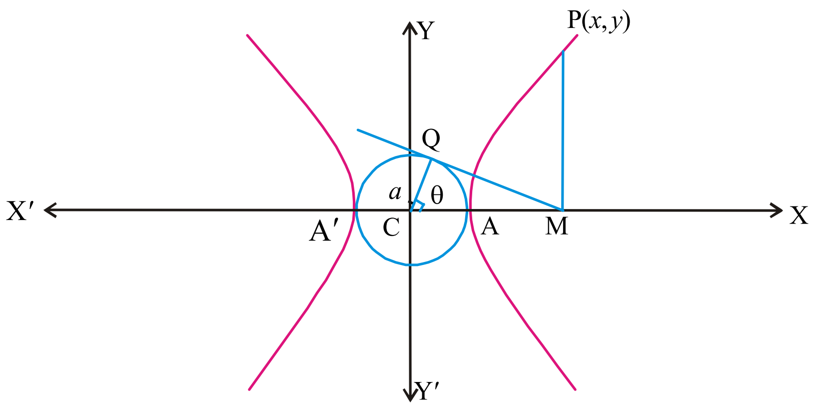
$\therefore$ The eccentricity of a rectangular hyperbola is $\sqrt{2}$.

Definition: Auxiliary Circle:

The circle described on the transverse axis of a hyperbola as diameter is called as the auxiliary circle of the hyperbola.

The equation of the auxiliary circle of the hyperbola S = 0 is $x^2 + y^2 = a^2$.

Parametric equations:



Let the equation of the hyperbola be S = 0, then the equation of the auxiliary circle is $x^2+y^2=a^2$.

Let $P(x, y)$ be any point on the hyperbola and C be the centre.

Let M be the projection of P on the transverse axis. Draw the tangent QM to the auxiliary circle from M. Let $\angle MCQ = \Theta$

Then $\boxed{\begin{array}{l} x = a\sec\Theta \\ y = b\tan\Theta \end{array}}$ are the parametric equations of the hyperbola S = 0.

$$\Theta \in [0, 2\pi),\ \Theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$$

Definition: Conjugate Hyperbola

The hyperbola whose transverse and conjugate axis are respectively the conjugate and transverse axis of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

If S = $\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$ and $S' = \frac{x^2}{a^2}-\frac{y^2}{b^2}+1=0$ then each hyperbola is the conjugate of the other.

PROBLEMS

Very Short Answer Type Questions

1. One focus of a hyperbola is located at the point (1, -3) and the corresponding directrix is the line y = 2, Find the equation of the hyperbola if its eccentricity is $\frac{3}{2}$.

Sol. **Note :** To find the equation of the conic (parabola, ellipse, hyperbola), when eccentricity, focus and directrix are given, always use the definition of conic ie. $\frac{SP}{PM}=e$.

Let $S=(1,-3)$, $e=\frac{3}{2}$, directrix is $y-2=0$

Let $P(x_1, y_1)$ be any point on the hyperbola.

Then according to the definition of hyperbola, $\frac{SP}{PM}=e$

Where PM is the perpendicular distance from P to the directrix

$\Rightarrow$ SP = e PM

$\Rightarrow \sqrt{(x_1-1)^2+(y_1+3)^2} = \frac{3}{2}\left|\frac{y_1-2}{\sqrt{0^2+1^2}}\right|$

$\left[\because \text{ Formula:- } \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}} \right.$

Here $ax+by+c=0$

That is $\left. o.x+1.y-2=0 \right]$

Squaring on both sides, we get

$(x_1-1)^2+(y_1+3)^2 = \frac{9}{4}(y_1-2)^2$

$\Rightarrow x_1^2+1-2x_1+y_1^2+9+6y_1 = \frac{9}{4}(y_1^2+4-4y_1)$

$\Rightarrow 4(x_1^2+y_1^2-2x_1+6y_1+10) = 9y_1^2+36-36y_1$

$\Rightarrow 4x_1^2+4y_1^2-8x_1+24y_1+40-9y_1^2-36+36y_1=0$

$\Rightarrow 4x_1^2-5y_1^2-8x_1+60y_1+4=0$

$\therefore$ The locus of $P(x_1 y_1)$ is $4x^2-5y^2-8x+60y+4=0$ which is the required Hyperbola.

2. If the eccentricity of a hyperbola is $\frac{5}{4}$, then find the eccentricity of the congujate hyperbola.

Sol. We know that

If e and e' are the eccentricities of a hyperbola and its conjugate hyperbola, then $\frac{1}{e^2}+\frac{1}{(e')^2}=1$

Given $e=\frac{5}{4}$

$$\Rightarrow \frac{1}{\left(\frac{5}{4}\right)^2}+\frac{1}{(e')^2}=1$$

$$\Rightarrow \frac{1}{(e')^2}=1-\frac{16}{25}=\frac{9}{25}$$

$$\Rightarrow \frac{(e')^2}{1}=\frac{25}{9} \Rightarrow e'=\sqrt{\frac{25}{9}}=\frac{5}{3}$$

$\therefore e'=\frac{5}{3}$ = eccentricity of the conjugate hyperbola.

Short Answer Type Questions

1. If e and e_1 are the eccentricities of a hyperbola and its conjugate hyperbola, then prove that $\frac{1}{e^2}+\frac{1}{e_1^2}=1$

Sol.: Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ___ (1)

Its eccentricity 'e' is given by $b^2=a^2(e^2-1) \Rightarrow \frac{b^2}{a^2}=e^2-1$

$$\Rightarrow e^2=1+\frac{b^2}{a^2} \Rightarrow e^2=\frac{a^2+b^2}{a^2}$$

$$\Rightarrow \frac{1}{e^2}=\frac{a^2}{a^2+b^2} \quad \text{___ (2)}$$

The conjugate hyperbola of equation (1) is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

Its eccentricity e_1 is given by, $a^2=b^2(e_1^2-1)$

$$\Rightarrow e_1^2-1=\frac{a^2}{b^2}$$

$$\Rightarrow e_1^2=1+\frac{a^2}{b^2}=\frac{b^2+a^2}{b^2}$$

$$\Rightarrow \frac{1}{e_1^2}=\frac{b^2}{a^2+b^2} \quad \text{__(3)}$$

From (2) and (3) we get

$$\frac{1}{e^2}+\frac{1}{e_1^2}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}$$

$$=\frac{a^2+b^2}{a^2+b^2}=1$$ Hence proved.

2. Find the centre, foci, eccentricity, equation of the directrices, length of the latus rectum of the following hyperbolas,

(i) $16y^2-9x^2=144$

(ii) $9x^2-16y^2+72x-32y-16=0$

Sol. (i) The given hyperbola is $16y^2-9x^2=144$

$$\Rightarrow 9x^2-16y^2=-144$$

$$\Rightarrow \frac{9x^2}{144} - \frac{16y^2}{144} = \frac{-144}{144}$$

$$\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = -1$$

It is a hyperbola whose centre is (0, 0)

and transverse axis is along$y$-axis.

So, $a^2 = b^2(e^2 - 1)$, where $a^2 = 16, b^2 = 9$

$\Rightarrow 16 = 9(e^2 - 1)$ $\quad a = 4, \quad b = 3$

$$\Rightarrow e^2 - 1 = \frac{16}{9}$$

$$\Rightarrow e^2 = \frac{16}{9} + 1 = \frac{25}{9}$$

$$\Rightarrow \boxed{e = \frac{5}{3}}$$

$\therefore$ Centre = (0, 0)

Foci $= \left(0, \pm\, be\right) = \left(0, \pm\, 5\right)$

Eccentricity, e = $\frac{5}{3}$

Length of latus rectum = $\frac{2a^2}{b} = \frac{32}{3}$

Equation of the directrices = $y = \pm \frac{b}{e}$

$$\Rightarrow y = \pm\, \frac{9}{5}$$

$$\Rightarrow 5y \pm 9 = 0$$

(ii) Given hyperbola is $9x^2 - 16y^2 + 72x - 32y - 16 = 0$

$$\Rightarrow (9x^2 + 72x) - (16y^2 + 32y) - 16 = 0$$

$$\Rightarrow 9(x^2 + 8x) - 16(y^2 + 2y) - 16 = 0$$

$$\Rightarrow 9[x^2 + 2.x.4] - 16[y^2 + 2.y.1] = 16$$

$$\Rightarrow 9[x^2 + 2.x.4 + 4^2 - 4^2] - 16[y^2 + 2.y.1 + 1^2 - 1^2] = 16$$

$$\Rightarrow 9[(x+4)^2 - 16] - 16[(y+1)^2 - 1] = 16$$

$$\Rightarrow 9(x+4)^2 - 144 - 16(y+1)^2 + 16 = 16$$

$$\Rightarrow 9(x+4)^2 - 16(y+1)^2 = 144$$

$$\Rightarrow \frac{9(x+4)^2}{144} - \frac{16(y+1)^2}{144} = 1$$

$$\Rightarrow \frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$$

Comparing this equation with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

we get, $h = -4,\ k = -1,\ a = 4,\ b = 3,\ b^2 = a^2(e^2-1)$

$$\Rightarrow e^2 - 1 = \frac{b^2}{a^2} = \frac{9}{16}$$

$$\Rightarrow e^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\Rightarrow e = \frac{5}{4}$$

$\therefore$ Centre = $(h, k) = (-4, -1)$

Foci = $(h \pm ae, k) = (-4 \pm 5, -1)$

= $(-4+5, -1)$ and $(-4-5, -1)$

= $(1, -1)$ and $(-9, -1)$

Eccentricity $e = \frac{5}{4}$

Equations of directrices : $x - h = \pm \frac{a}{e}$

$$\Rightarrow x + 4 = \pm \frac{16}{5}$$

$\Rightarrow 5x + 20 = \pm 16$

$\Rightarrow 5x + 20 = 16$ and $5x + 20 = -16$

$\Rightarrow 5x + 4 = 0$ and $5x + 36 = 0$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2}$

3. Find the equation to the hyperbola whose foci are (4, 2) and (8, 2) and eccentricity is 2.

Sol.: The foci of the hyperbola are S = (4, 2) and $S' = (8, 2)$. Since the y-cordinate of S & S' are same, $\overleftrightarrow{SS'}$ is parallel to X-axis

$\Rightarrow$ The transverse axis is parallel to X-axis.

$\Rightarrow$ The equation of hyperbola is of the form,

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \qquad \text{\_\_\_ (1)}$$

Centre = C = (h,k) = Midpoint of $SS'=\left(\frac{4+8}{2},\frac{2+2}{2}\right)$

$$=(6,2)$$

$$\Rightarrow h=6, k=2$$

Distance between foci = SS' = $\sqrt{(8-4)^2+(2-2)^2}=4$

e = 2 (given) $\Rightarrow 2ae=4$

$\Rightarrow 2.a.2=4$

$\Rightarrow a=1$

$$b^2=a^2(e^2-1)$$

$$=1(4-1)=3$$

$\therefore$ The hyperbola is

$$\frac{(x-6)^2}{1^2}-\frac{(y-2)^2}{3}=1$$

$$\Rightarrow \frac{3(x-6)^2-(y-2)^2}{3}=1$$

$$\Rightarrow 3(x^2+36-12x)-(y^2+4-4y)=3$$

$$\Rightarrow 3x^2-y^2-36x+4y+101=0$$

4. Find the equation of the hyperbola of given length of transverse axis 6 whose vertex bisects the distance between the centre and the focus.

Sol.: Let the hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Length of transverse axis = $2a=6$ (given)

$$\Rightarrow \boxed{a=3}$$

Vertex bisects the distance between the centre & focus

$\Rightarrow$ Vertex is the midpoint of C, S

Where C = (0, 0), focus = S = $(ae, 0)$

Vertex = $(a, 0)$

$$\Rightarrow (a,0)=\left(\frac{0+ae}{2},\frac{0+0}{2}\right)=\left(\frac{ae}{2},0\right)$$

$$\Rightarrow a=\frac{ae}{2}\Rightarrow \boxed{e=2}$$

Now $b^2 = a^2(e^2-1) = 9(4-1) = 27$.

Hence the required equation of hyperbola is, $\dfrac{x^2}{9} - \dfrac{y^2}{27} = 1$

$$\frac{3x^2 - y^2}{27} = 1 \Rightarrow 3x^2 - y^2 = 27$$

5. If the lines $3x - 4y = 12$ and $3x + 4y = 12$ meets on a hyperbola S = 0, then find the eccentricity of the hyperbola S = 0.

Sol.: The lines $3x - 4y = 12$ and $3x + 4y = 12$ meet on the hyperbola S = 0.

The combined equation of the lines is $(3x-4y)(3x+4y) = 12 \times 12$

$$\Rightarrow 9x^2 - 16y^2 = 144$$

$$\Rightarrow \frac{9x^2}{144} - \frac{16y^2}{144} = \frac{144}{144}$$

$$\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$$ which represent a hyperbola.

$\therefore b^2 = a^2(e^2-1)$ where $a^2 = 16$, $b^2 = 9$

$$\Rightarrow 9 = 16(e^2-1)$$

$$\Rightarrow \frac{9}{16} = e^2 - 1 \quad \Rightarrow e^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\Rightarrow e = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

Unit 6

Integration

Integration is the inverse process of differentiation. The process of finding the function whose derivative is given, is called as Integration.

Definition: Let E be a subset of **R** such that E contains a right or a left neighbourhood of each of its points and let f: E $\rightarrow$ R be a function. If there is a function F on E such that F'(x) = $f(x)$ $\forall x \in$ E, then we call F an **antiderivative of f or a primitive of f.**

Indefinite Integral: Let f: I $\rightarrow$ R. Suppose that f has an antiderivate F on I. Then we say that f has an integral on I and for any real constant c, we call F + c an indefinite integral of f over I, denote it by $\int f(x)dx$ and read it as 'integral $f(x)$ dx'. We also denote $\int f(x)dx$ as $\int f$.

Thus we have $\int f = \int f(x)dx = \mathrm{F}(x) + c$.

'c' is called a 'constant of integration'.

'f' is called the 'integrand' and 'x' is called the 'variable of integration'.

Note: (i) $\frac{d}{dx}\left[f(x)dx\right] = f(x)$

(ii) $\int f'(x)\,dx = f(x) + c$, 'c' is the constant of integration.

$$\int \frac{d}{dx} f(x)dx = f(x) + c$$

(iii) $\frac{d}{dx}\left[f(x)+c\right] = g(x) \Rightarrow \int g(x)dx = f(x) + c \Rightarrow \int \frac{d}{dx}\left[f(x)+c\right]dx = f(x) + c$

(iv) $y = f(x) \Rightarrow dy = f'(x)dx$

Standard Formulae

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$ $\qquad$ $\int dx = \int 1.dx = x + c$

$\int x.dx = \frac{x^2}{2} + c$ $\qquad$ $\int \sqrt{x}\, dx = \frac{x^{3/2}}{3/2} + c$

$\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$ $\qquad$ $\int x^2\, dx = \frac{x^3}{3} + c$

$\int x^3\, dx = \frac{x^4}{4} + c$

2. $\int \frac{1}{x} dx = \log_e |x| + c$ $\qquad$ 3. $\int a^x .dx = \frac{a^x}{\log_e a} + c,\ a > 0,\ a \neq 1$

4. $\int e^x dx = e^x + c$ $\qquad$ 5. $\int \sin x\, dx = -\cos x + c$

6. $\int \cos x\, dx = \sin x + c$ $\qquad$ 7. $\int \sec^2 x\, dx = \tan x + c$

8. $\int \mathrm{cosec}^2 x\, dx = -\cot x + c$ $\qquad$ 9. $\int \sec x \tan x\, dx = \sec x + c$

10. $\int \mathrm{cosec}\, x \cot x\, dx = -\mathrm{cosec}\, x + c$

Examples

(i) $d(x^2) = 2x\, dx$ $\qquad$ (ii) $d(t^2) = 2t\, dt$

(iii) $d(x^3 y^3) = x^3.3y^2\, dy + y^3.3x^2\, dx$ $\qquad$ (iv) $d\left(\frac{x^3}{y^3}\right) = \frac{y^3.3x^2\, dx - x^3.3y^2\, dy}{\left(y^3\right)^2}$

11. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c = -\cos^{-1} x + c$ $\qquad$ $\left(\because \cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}\right)$

12. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c = -\cot^{-1} x + c$ $\qquad$ $\left(\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right)$

13. $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1} x + c = -\mathrm{cosec}^{-1} x + c$ $\qquad$ $\left(\because \sec^{-1} x + \mathrm{cosec}^{-1} x = \frac{\pi}{2}\right)$

14. $\int \sinh x\, dx = \cosh x + c$ $\qquad$ 15. $\int \cosh x = \sinh x + c$

16. $\int \mathrm{sech}^2 x\, dx = \tanh x + c$ $\qquad$ 17. $\int \mathrm{cosech}^2 x = -\coth x + c$

18. $\int \mathrm{sech}\, x.\tanh x\, dx = -\mathrm{sech}\, x + c$ $\qquad$ 19. $\int \mathrm{cosech}\, x.\coth x\, dx = -\mathrm{cosech}\, x + c$

20. $\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + c = \log_e \left[x + \sqrt{x^2+1}\right] + c$

21. $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + c = \log_e \left|x + \sqrt{x^2-1}\right| + c$

22. $\int \frac{1}{1-x^2} dx = \tanh^{-1} x + c = \coth^{-1} x + c$

23. $\int (f+g)(x)\, dx = \int f(x).dx + \int g(x).dx + c$

24. $\int a.f(x)\, dx = a\int f(x).dx + c$ where $a \in \mathbf{R}$

Integration by the method of Substitution

Formulae

1. $\int f'[g(x)].g'(x).dx = f[g(x)] + c$

2. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

3. $\int [f(x)]^n .f'(x).dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$

4. $\int f(x).f'(x).dx = \frac{[f(x)]^2}{2} + c$

5. $\int \sqrt{f(x)}.f'(x).dx = \frac{[f(x)]^{3/2}}{3/2} + c$

6. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

7. $\int f'(ax+b)\, dx = \frac{f(ax+b)}{a} + c$

8. $\int \frac{f'(x)}{f^2(x)} dx = \frac{-1}{f(x)} + c$

9. $\int \tan x\, dx = \log|\sec x| + c = -\log|\cos x| + c$

10. $\int \cot x\, dx = \log|\sin x| + c$

11. $\int \sec x\, dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c\right|$

12. $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + c = \log\left|\tan\left(\frac{x}{2}\right)\right| + c = -\log|\operatorname{cosec} x + \cot x| + c$

13. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

14. $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

15. $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

16. $\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + c = \log\left|\frac{x+\sqrt{a^2+x^2}}{a}\right| + c$

17. $\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

18. $\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + c = \log\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c$

19. $\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + c$

20. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

21. $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$

Examples:

1. $\int \frac{e^x}{e^x + 1}\,dx = \log\left|e^x + 1\right| + c$

2. $\int \frac{1}{ax+b}\,dx = \frac{\log|ax+b|}{a} + c$, $\int \frac{1}{\sqrt{ax+b}}\,dx = \frac{2\sqrt{ax+b}}{a} + c$, $\int \frac{1}{3-8x}\,dx = \frac{\log|3-8x|}{-8} + c$

3. $\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$, $\int e^{-x}\,dx = \frac{e^{-x}}{-1} + c$

4. $\int \sin(ax+b)\,dx = \frac{-\cos(ax+b)}{a} + c$, $\int \sin(9x)\,dx = \frac{-\cos(9x)}{9} + c$

5. $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + c$, $\int \cos(2x)\,dx = \frac{\sin(2x)}{2} + c$

6. $\int (2+3x)^n\,dx = \frac{\frac{(2+3x)^{n+1}}{n+1}}{3} + c$, $\int (2+3x)^4\,dx = \frac{\frac{(2+3x)^5}{5}}{3} + c$

7. $\int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a} + c$

8. $\int \operatorname{cosec}^2(ax+b)\,dx = \dfrac{-\cot(ax+b)}{a} + c$

9. $\int \operatorname{cosec}(ax+b).\cot(ax+b)\,dx = \dfrac{-\operatorname{cosec}(ax+b)}{a} + c$

10. $\int \sec(ax+b).\tan(ax+b)\,dx = \dfrac{\sec(ax+b)}{a} + c$

11. $\int \sqrt{7-5x}\,dx = \dfrac{\dfrac{(7-5x)^{3/2}}{3/2}}{-5} + c$ $\left[\because \int \sqrt{x}\,dx = \dfrac{x^{3/2}}{3/2}\right]$

12. $\int \dfrac{1}{\sqrt{3-9x}}\,dx = \dfrac{2\sqrt{3-9x}}{-9} + c$ $\left[\because \int \dfrac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\right]$

13. $\int \dfrac{1}{4-\dfrac{5x}{7}}\,dx = \dfrac{\log\left|4-\dfrac{5x}{7}\right|}{-\dfrac{5}{7}} + c$ $\left[\because \int \dfrac{1}{x}\,dx = \log x\right]$

14. $\int e^{3-\frac{2x}{5}}\,dx = \dfrac{e^{3-\frac{2x}{5}}}{-\dfrac{2}{5}} + c$ $\left[\because \int e^x\,dx = e^x\right]$

15. $\int \dfrac{1}{1+x}\,dx = \log|1+x| + c,$ $\int \dfrac{1}{1+x^2}\,dx = \tan^{-1}x + c$ (understand the difference)

Solved Problems

1. Find $\int \cot^2 x\,dx$.

Sol. $\int \cot^2 x\,dx = \int \left(\operatorname{cosec}^2 x - 1\right)dx$

$= \int \operatorname{cosec}^2 x.dx - \int 1.dx = -\cot x - x + c$

2. Find $\int \left(\dfrac{x^6-1}{1+x^2}\right)dx$.

Sol. $\therefore \int \left(\dfrac{x^6-1}{1+x^2}\right)dx = \int \left\{(x^4-x^2+1) + \dfrac{-2}{1+x^2}\right\}dx$

$= \dfrac{x^5}{5} - \dfrac{x^3}{3} + x - 2\tan^{-1}x + c$

3. Find $\int (1-x)(4-3x)(3+2x)\,dx$.

Sol. $(1-x)(4-3x)(3+2x) = (1-x)\left(12+8x-9x-6x^2\right)$

$= (1-x)\left(12-x-6x^2\right) = 12-x-6x^2-12x+x^2+6x^3 = 6x^3-5x^2-13x+12$

$\therefore \int (1-x)(4-3x)(3+2x)\,dx = \int (6x^3-5x^2-13x+12)\,dx$

$$= 6\frac{x^4}{4} - 5\frac{x^3}{3} - 13\frac{x^2}{2} + 12x + c = \frac{3x^4}{2} - \frac{5x^3}{3} - \frac{13x^2}{2} + 12x + c$$

4. Find $\int \sqrt{1+\sin 2x}\,dx$.

Sol. $\int \sqrt{1+\sin 2x}\,dx = \int \sqrt{1+2\sin x\cos x}\,dx$

$= \int \sqrt{(\sin^2 x+\cos^2 x)+2\sin x\cos x}\,dx = \int \sqrt{(\sin x+\cos x)^2}\,dx$

$= \int (\sin x+\cos x)\,dx = -\cos x + \sin x + c$

5. Evaluate $\int \frac{2x^3-3x+5}{2x^2}dx$ for $x > 0$ and verify the result by differentiation.

Sol. $\int \frac{2x^3-3x+5}{2x^2}dx = \int \left(\frac{2x^3}{2x^2} - \frac{3x}{2x^2} + \frac{5}{2x^2}\right)dx$

$$= \int \left(x - \frac{3}{2}.\frac{1}{x} + \frac{5}{2}x^{-2}\right)dx$$

$$= \frac{x^2}{2} - \frac{3}{2}\log|x| + \frac{5}{2}.\frac{x^{-2+1}}{-2+1} + c$$

$$= \frac{x^2}{2} - \frac{3}{2}\log|x| + \frac{5}{2}.\frac{x^{-1}}{-1} + c$$

$$= \frac{x^2}{2} - \frac{3}{2}\log|x| - \frac{5}{2}.\frac{1}{x} + c$$

Verification:

$$\frac{d}{dx}\left[\frac{x^2}{2} - \frac{3}{2}\log|x| - \frac{5}{2}.\frac{1}{x} + c\right]$$

$$= \frac{2x}{2} - \frac{3}{2}.\frac{1}{x} - \frac{5}{2}(-x^{-1-1}) = x - \frac{3}{2x} + \frac{5}{2x^2}$$

$= \frac{x(2x^2)-3(x)+5}{2x^2} == \frac{2x^3-3x+5}{2x^2}$. Hence verified.

6. Evaluate $\int \frac{x^2+3x-1}{2x}dx$.

Sol. $\int \frac{x^2+3x-1}{2x}dx = \int \left(\frac{x^2}{2x}+\frac{3x}{2x}-\frac{1}{2x}\right)dx$

$$= \int\left(\frac{1}{2}.x+\frac{3}{2}.1-\frac{1}{2}.\frac{1}{x}\right)dx = \frac{1}{2}\int x.dx+\frac{3}{2}\int 1.dx-\frac{1}{2}\int\frac{1}{x}.dx$$

$$= \frac{1}{2}.\frac{x^2}{2}+\frac{3}{2}x-\frac{1}{2}\log|x|+c = \frac{x^2}{4}+\frac{3x}{2}-\frac{1}{2}\log|x|+c$$

7. Evaluate $\int\left(1+\frac{2}{x}-\frac{3}{x^2}\right)dx$.

Sol. $\int\left(1+\frac{2}{x}-\frac{3}{x^2}\right)dx = x+2\log|x|-3.\frac{x^{-2+1}}{-2+1}+c = x+2\log|x|+3.\frac{1}{x}+c$

8. Evaluate $\int\left(x+\frac{4}{1+x^2}\right)dx$.

Sol. $\int\left(x+\frac{4}{1+x^2}\right)dx = \int x.dx+4\int\frac{1}{1+x^2}dx = \frac{x^2}{2}+4\tan^{-1}x+c$

9. Evaluate $\int\left(e^x-\frac{1}{x}+\frac{2}{\sqrt{x^2-1}}\right)dx$.

Sol. $\int\left(e^x-\frac{1}{x}+\frac{2}{\sqrt{x^2-1}}\right)dx = e^x-\log|x|+2\cosh^{-1}x+c$

10. Evaluate $\int\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)dx$.

Sol. $\int\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)dx = \tanh^{-1}x+\tan^{-1}x+c$

11. Evaluate $\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{2}{\sqrt{1+x^2}}\right)dx$.

Sol. $\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{2}{\sqrt{1+x^2}}\right)dx = \int\frac{1}{\sqrt{1-x^2}}dx+2\int\frac{1}{\sqrt{1+x^2}}dx$

$$= \sin^{-1}x+2\sinh^{-1}x+c$$

12. Evaluate $\int e^{\log(1+\tan^2 x)}\, dx$.

Sol. $\int e^{\log(1+\tan^2 x)}\, dx = \int e^{\log(\sec^2 x)}\, dx = \int \sec^2 x\, dx = \tan x + c \qquad \left(\because a^{\log_a x} = x\right)$

13. Evaluate $\int \frac{\sin^2 x}{1+\cos 2x} dx$.

Sol. $\int \frac{\sin^2 x}{1+\cos 2x} dx = \int \frac{\sin^2 x}{2\cos^2 x} dx = \frac{1}{2}\int \tan^2 x\, dx = \frac{1}{2}\int (\sec^2 x - 1)\, dx$

$$= \frac{1}{2}\int \sec^2 x\, dx - \frac{1}{2}\int 1.dx = \frac{1}{2}\tan x - \frac{1}{2}x + c$$

14. Evaluate $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right) dx$.

Sol. $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right) dx = 3\int \frac{1}{\sqrt{x}} dx - 2\int \frac{1}{x} dx + \frac{1}{3}\int x^{-2}\, dx$

$$= 3.2\sqrt{x} - 2\log|x| + \frac{1}{3}\frac{x^{-2+1}}{(-2+1)} + c = 6\sqrt{x} - 2\log|x| - \frac{1}{3}.\frac{1}{x} + c$$

15. Evaluate $\int \left(\frac{\sqrt{x}+1}{x}\right)^2 dx$.

Sol. $\int \left(\frac{\sqrt{x}+1}{x}\right)^2 dx = \int \frac{x+1+2\sqrt{x}}{x^2} dx$

$$= \int \left(\frac{x}{x^2} + \frac{1}{x^2} + \frac{2x^{1/2}}{x^2}\right) dx = \int \left(\frac{1}{x} + x^{-2} + 2.x^{\frac{1}{2}-2}\right) dx$$

$$= \int \left(\frac{1}{x} + x^{-2} + 2.x^{\frac{-3}{2}}\right) dx = \log|x| + \frac{x^{-2+1}}{-2+1} + 2.\frac{x^{\frac{-3}{2}+1}}{-\frac{3}{2}+1} + c$$

$$= \log|x| - \frac{1}{x} - 4.x^{-\frac{1}{2}} + c = \log|x| - \frac{1}{x} - \frac{4.}{\sqrt{x}} + c$$

16. Evaluate $\int \left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right) dx$.

Sol. $\int \left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right) dx = 2\sqrt{x} + 2\cosh^{-1} x - \frac{3}{2}\left(-\frac{1}{x}\right)$

$$= 2\sqrt{x} + 2\cosh^{-1} x + \frac{3}{2x} + c$$

17. Evaluate $\int \left(\cosh x + \frac{1}{\sqrt{x^2+1}} \right) dx$.

Sol. $\int \left(\cosh x + \frac{1}{\sqrt{x^2+1}} \right) dx = \sinh x + \sinh^{-1} x + c$

18. Evaluate $\int \left(\sinh x + \frac{1}{\left(x^2-1\right)^{1/2}} \right) dx$.

Sol. $\int \left(\sinh x + \frac{1}{\left(x^2-1\right)^{1/2}} \right) dx = \int \sinh x\, dx + \int \frac{1}{\sqrt{x^2-1}} dx$

$$= \cosh x + \cosh^{-1} x + c$$

19. Evaluate $\int \frac{(a^x - b^x)^2}{a^x b^x} dx$.

Sol. $\int \frac{(a^x - b^x)^2}{a^x b^x} dx = \int \frac{a^{2x} + b^{2x} - 2a^x b^x}{a^x b^x} dx$

$$= \int \left(\frac{a^{2x}}{a^x b^x} + \frac{b^{2x}}{a^x b^x} - \frac{2a^x b^x}{a^x b^x} \right) dx$$

$$= \int \left(\frac{a^x}{b^x} + \frac{b^x}{a^x} - 2 \right) dx \ = \int \frac{a^x}{b^x} dx + \int \frac{b^x}{a^x} dx - 2\int 1.dx$$

$$= \frac{\left(\frac{a}{b}\right)^x}{\log_e\left(\frac{a}{b}\right)} + \frac{\left(\frac{b}{a}\right)^x}{\log_e\left(\frac{b}{a}\right)} - 2x + c$$

20. Evaluate $\int \sec^2 x \operatorname{cosec}^2 x\, dx$.

Sol. $\int \sec^2 x \operatorname{cosec}^2 x\, dx = \int (1 + \tan^2 x)(\operatorname{cosec}^2 x)\, dx$

$$= \int (\operatorname{cosec}^2 x + \tan^2 x \operatorname{cosec}^2 x)\, dx$$

$$= \int (\operatorname{cosec}^2 x + \sec^2 x)\, dx$$

$$= -\cot x + \tan x + c$$

Alternate method:

$$\int \sec^2 x \operatorname{cosec}^2 x\, dx = \int \frac{1}{\cos^2 x} . \frac{1}{\sin^2 x} dx$$

$$= \int \frac{1}{\cos^2 x.\sin^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\cos^2 x.\sin^2 x} dx \qquad \left[\because \sin^2 x + \cos^2 x = 1\right]$$

$$= \left(\int \frac{\sin^2 x}{\cos^2 x.\sin^2 x} + \frac{\cos^2 x}{\cos^2 x.\sin^2 x} \right) dx$$

$$= \left(\int \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx$$

$$= \left(\int \sec^2 x + \operatorname{cosec}^2 x \right) dx$$

$$= \tan x - \cot x + c$$

21. Evaluate $\int \frac{1+\cos^2 x}{1-\cos 2x} dx$.

Sol. $\int \frac{1+\cos^2 x}{1-\cos 2x} dx = \int \frac{1+\cos^2 x}{2\sin^2 x} dx$

$$= \int \left(\frac{1}{2\sin^2 x} + \frac{\cos^2 x}{2\sin^2 x} \right) dx = \int \left(\frac{1}{2}\operatorname{cosec}^2 x + \frac{1}{2}\cot^2 x \right) dx$$

$$= \frac{1}{2}\int \left(\operatorname{cosec}^2 x + \cot^2 x \right) dx = \frac{1}{2}\int \left(\operatorname{cosec}^2 x + \operatorname{cosec}^2 x - 1 \right) dx$$

$$= \frac{1}{2}\int \left(2\operatorname{cosec}^2 x - 1 \right) dx = \frac{1}{2}\left[2(-\cot x) - x \right]$$

$$= -\cot x - \frac{x}{2} + c$$

22. Evaluate $\int \sqrt{1-\cos 2x}\, dx$.

Sol. $\int \sqrt{1-\cos 2x}\, dx = \int \sqrt{2\sin^2 x}\, dx$

$$= \int \sqrt{2}\sin x\, dx = \sqrt{2}(-\cos x) = -\sqrt{2}\cos x + c$$

23. Evaluate $\int \frac{1}{\cosh x + \sinh x} dx$.

Sol. $\int \frac{1}{\cosh x + \sinh x} dx = \int \frac{\cosh^2 x - \sinh^2 x}{\cosh x + \sinh x} dx \qquad \left(\because \cosh^2 x - \sinh^2 x = 1\right)$

$$= \int \frac{(\cosh x + \sinh x)(\cosh x - \sinh x)}{\cosh x + \sinh x} dx$$

$$= \int (\cosh x - \sinh x)\, dx = \sinh x - \cosh x + c$$

24. Evaluate $\int \frac{1}{1+\cos x} dx$.

Sol. $\int \frac{1}{1+\cos x} dx = \int \frac{1}{1+\cos x} \times \frac{1-\cos x}{1-\cos x} dx$

$$= \int \frac{1-\cos x}{1-\cos^2 x} dx = \int \frac{1-\cos x}{\sin^2 x} dx$$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin x \sin x} \right) dx$$

$$= \int \left(\text{cosec}^2 x - \cot x . \text{cosec}\, x \right) dx$$

$$= -\cot x + \text{cosec}\, x + c$$

Note: To evaluate $\int \frac{1}{1-\cos x} dx$, $\int \frac{1}{1-\sin x} dx$, $\int \frac{1}{1+\sin x} dx$ similar method can be used.

Integration by Substitution

Evaluate the following integrals:

1. $\int \frac{e^x}{e^x+1} dx$

Sol. Put $e^x + 1 = t \Rightarrow e^x.dx = dt.$

$$\int \frac{e^x}{e^x+1} dx = \int \frac{dt}{t} = \int \frac{1}{t} dt$$

$$= \log |t| = \log |e^x + 1| + c$$

(OR) $\int \frac{f'(x)}{f(x)} dx = \log |f(x)|$

Let $f(x) = e^x + 1 \Rightarrow f'(x) = e^x$

$$\therefore \int \frac{e^x}{e^x+1} dx = \int \frac{f'(x)}{f(x)} dx = \log |f(x)| = \log |e^x + 1| + c$$

2. $\int \frac{x^2}{\sqrt{1-x}} dx$

Sol. Put $\sqrt{1-x} = t \Rightarrow 1 - x = t^2 \Rightarrow -dx = 2t\,dt \Rightarrow x = 1 - t^2$

$$\therefore \int \frac{x^2}{\sqrt{1-x}} dx = \int \frac{(1-t^2)^2}{t} (-2t) dt$$

$$= -2\int (1-t^2)^2\, dt = -2\int (1+t^4-2t^2)\, dt = -2\left[t + \frac{t^5}{5} - \frac{2t^3}{3} \right] = -2t - \frac{2}{5}t^5 + \frac{4}{3}t^3$$

$$= -2\sqrt{1-x} - \frac{2}{5}(\sqrt{1-x})^5 + \frac{4}{3}(\sqrt{1-x})^3 + c$$

3. $\int \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}} dx$

Sol. Put $\sin^{-1}(x) = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

$$\therefore \int (\sin^{-1} x)^2 . \frac{1}{\sqrt{1-x^2}} dx = \int t^2 \, dt = \frac{t^3}{3} = \frac{(\sin^{-1} x)^3}{3} + c \quad \text{(or)}$$

$$\int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1}, \quad \text{where } f(x) = \sin^{-1}(x), \; f'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \int (\sin^{-1} x)^2 . \frac{1}{\sqrt{1-x^2}} dx = \frac{(\sin^{-1} x)^{2+1}}{2+1} = \frac{(\sin^{-1} x)^3}{3} + c$$

4. $\int \frac{1}{1+(2x+1)^2} dx$.

Sol. Put $2x+1 = t \Rightarrow 2.1.dx = dt \Rightarrow dx = \frac{dt}{2}$

$$\int \frac{1}{1+(2x+1)^2} dx = \int \frac{1}{1+t^2} \frac{dt}{2}$$

$$= \frac{1}{2} \int \frac{1}{1+t^2} dt = \frac{1}{2} \tan^{-1} t = \frac{1}{2} \tan^{-1}(2x+1) + c \quad \text{(or)}$$

$$\int f'(ax+b) \, dx = \frac{f(ax+b)}{a}$$

$$\therefore \int \frac{1}{1+x^2} dx = \tan^{-1} x \Rightarrow \int \frac{1}{1+(ax+b)^2} dx = \frac{\tan^{-1}(ax+b)}{a}$$

$$\Rightarrow \int \frac{1}{1+(2x+1)^2} dx = \frac{\tan^{-1}(2x+1)}{2} = \frac{1}{2} \tan^{-1}(2x+1) + c$$

5. $\int \frac{x^5}{1+x^{12}} dx$.

Sol. Put $x^6 = t \Rightarrow 6.x^5 dx = dt \Rightarrow x^5 dx = \frac{dt}{6}$

$$\int \frac{x^5}{1+x^{12}} dx = \int \frac{x^5 dx}{1+(x^6)^2} dx = \int \frac{\frac{dt}{6}}{1+t^2}$$

$$= \frac{1}{6} \int \frac{1}{1+t^2} dt = \frac{1}{6} \tan^{-1} t = \frac{1}{6} \tan^{-1}(x^6) + c$$

6. $\int \cos^3 x \sin x\, dx$.

Sol. Put $\cos x = t \Rightarrow -\sin x dx = dt \Rightarrow \sin x dx = -dt$

$$\therefore \int \cos^3 x \sin x\, dx = \int t^3(-dt)$$

$$= -\int t^3 dt = -\left(\frac{t^4}{4}\right) = -\frac{(\cos x)^4}{4} = -\frac{\cos^4 x}{4} + c$$

7. $\int \left(1-\frac{1}{x^2}\right).e^{\left(x+\frac{1}{x}\right)} dx$

Sol. Put $x + \frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right) dx = dt$

$$\therefore \int \left(1-\frac{1}{x^2}\right).e^{\left(x+\frac{1}{x}\right)} dx = \int e^{\left(x+\frac{1}{x}\right)}\left(1-\frac{1}{x^2}\right).dx = \int e^t.dt = e^t = e^{\left(x+\frac{1}{x}\right)} + c$$

8. $\int \frac{1}{\sqrt{\sin^{-1} x}\sqrt{1-x^2}} dx$

Sol. Put $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} dx = dt$

$$\therefore \int \frac{1}{\sqrt{\sin^{-1} x}\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{\sin^{-1} x}}.\frac{1}{\sqrt{1-x^2}} dx$$

$$= \int \frac{1}{\sqrt{t}}.dt = 2\sqrt{t} = 2\sqrt{\sin^{-1} x} + c$$

9. $\int \frac{\sin^4 x}{\cos^6 x} dx$

Sol. $\int \frac{\sin^4 x}{\cos^6 x} dx = \int \frac{\sin^4 x}{\cos^4 x}.\frac{1}{\cos^2 x} dx$

$$= \int \tan^4 x.\sec^2 x\, dx \qquad \left[\because \int \left[f(x)\right]^n.f'(x).dx = \frac{\left[f(x)\right]^{n+1}}{n+1}\right]$$

$$= \frac{\tan^{4+1} x}{4+1} = \frac{\tan^5 x}{5} + c$$

10. $\int \sin^2 x\, dx$

Sol. $\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2} dx = \frac{1}{2}\int (1-\cos 2x)\, dx$

$$= \frac{1}{2}\left[\int 1.dx - \int \cos 2x.dx\right] = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$$

11. $\int \frac{x^2}{\sqrt{x+5}} dx$

Sol. Put $\sqrt{x+5} = t \Rightarrow x+5 = t^2 \Rightarrow dx = 2t\,dt$ and $x = t^2 - 5 \Rightarrow x^2 = (t^2-5)^2 = t^4 + 25 - 10t^2$

$$\therefore \int \frac{x^2}{\sqrt{x+5}} dx = \int \frac{t^4+25-10t^2}{t}.2t\,dt$$

$$= 2\int (t^4+25-10t^2).dt \quad = 2\left[\frac{t^5}{5} + 25t - \frac{10t^3}{3}\right]$$

$$= 2\left[\frac{\left(\sqrt{x+5}\right)^5}{5} + 25\sqrt{x+5} - \frac{10\left(\sqrt{x+5}\right)^3}{3}\right]$$

$$= \frac{2}{5}(x+5)^{5/2} + 50(x+5)^{1/2} - \frac{20}{3}(x+5)^{3/2} + c$$

12. $\int \frac{dx}{\sqrt{4-9x^2}} dx$

Sol. $\int \frac{1}{\sqrt{4-9x^2}} dx = \int \frac{1}{\sqrt{2^2-(3x)^2}} dx \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)\right]$

$$= \frac{\sin^{-1}\left(\frac{3x}{2}\right)}{3} = \frac{1}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c$$

13. $\int \frac{1}{1+4x^2} dx$

Sol. $\int \frac{1}{1+4x^2} dx = \int \frac{1}{1^2+(2x)^2} dx \qquad \left[\because \int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$

$$= \frac{\tan^{-1}\left(\frac{2x}{1}\right)}{2} = \frac{1}{2}\tan^{-1}(2x) + c$$

14. $\int \frac{1}{\sqrt{4-x^2}} dx$

Sol. $\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{2^2-x^2}} dx \quad = \sin^{-1}\left(\frac{x}{2}\right) + c \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)\right]$

15. $\int \sqrt{4x^2+9}\, dx$

Sol. $\int \sqrt{4x^2+9}\, dx = \int \sqrt{(2x)^2+3^2}\, dx$ $\left[\because \int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)\right]$

$$= \frac{\frac{2x}{2}\sqrt{4x^2+9} + \frac{9}{2}\sinh^{-1}\left(\frac{2x}{3}\right)}{2}$$

$$= \frac{x}{2}\sqrt{4x^2+9} + \frac{9}{4}\sinh^{-1}\left(\frac{2x}{3}\right) + c$$

16. $\int \sqrt{9x^2-25}\, dx$

Sol. $\int \sqrt{9x^2-25}\, dx = \int \sqrt{(3x)^2-5^2}\, dx$ $\left[\because \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)\right]$

$$= \frac{\frac{3x}{2}\sqrt{9x^2-25} - \frac{25}{2}\cosh^{-1}\left(\frac{3x}{5}\right)}{3}$$

$$= \frac{x}{2}\sqrt{9x^2-25} - \frac{25}{6}\cosh^{-1}\left(\frac{3x}{5}\right) + c$$

17. $\int \sqrt{16-25x^2}\, dx$

Sol. $\int \sqrt{16-25x^2}\, dx = \int \sqrt{4^2-(5x)^2}\, dx$ $\left[\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$

$$= \frac{\frac{5x}{2}\sqrt{16-25x^2} + \frac{16}{2}\sin^{-1}\left(\frac{5x}{4}\right)}{5}$$

$$= \frac{x}{2}\sqrt{16-25x^2} + \frac{16}{10}\sin^{-1}\left(\frac{5x}{4}\right) + c$$

18. $\int \frac{x}{1+x^2}\, dx$

Sol. $\int \frac{x}{1+x^2}\, dx = \frac{1}{2}\int \frac{2x}{1+x^2}\, dx$ $\left[\because \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$

$$= \frac{1}{2}\log|1+x^2| + c$$

19. $\int \frac{(\log x)^2}{x}\, dx$

Sol. $\int \frac{(\log x)^2}{x}\, dx = \int (\log x)^2 . \frac{1}{x} . dx$ $\left[\because \int [f(x)]^n . f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}\right]$

$$= \frac{(\log x)^{2+1}}{2+1} = \frac{(\log x)^3}{3} + c$$

20. $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

Sol. Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$$\int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^{\tan^{-1} x} . \frac{1}{1+x^2} . dx$$

$$= \int e^{t} . dt = e^{t} = e^{\tan^{-1} x} + c$$

21. $\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx$

Sol. Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$$\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx = \int \sin(\tan^{-1} x) . \frac{1}{1+x^2} dx$$

$$= \int \sin t . dt = -\cos t = -\cos(\tan^{-1} x) + c$$

22. $\int \frac{3x^2}{1+x^6} dx$

Sol. Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\int \frac{3x^2}{1+x^6} dx = \int \frac{3x^2 dx}{1+\left(x^3\right)^2} = \int \frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1}(x^3) + c$$

23. $\int \frac{2}{\sqrt{25+9x^2}} dx$

Sol. $\int \frac{2}{\sqrt{25+9x^2}} dx = 2\int \frac{1}{\sqrt{5^2+(3x)^2}} dx = 2 \frac{\sinh^{-1}\left(\frac{3x}{5}\right)}{3} = \frac{2}{3} \sinh^{-1}\left(\frac{3x}{5}\right) + c$

24. $\int \frac{3}{\sqrt{9x^2-1}} dx$

Sol. $\int \frac{3}{\sqrt{9x^2-1}} dx = 3\int \frac{1}{\sqrt{(3x)^2-1^2}} dx = 3 \frac{\cosh^{-1}\left(\frac{3x}{1}\right)}{3} = \cosh^{-1}(3x) + c$

25. $\int \sin mx \cos nx \, dx$

Sol. We have, $\sin mx \cos nx = \frac{1}{2}(2\sin mx \cos nx)$

$$= \frac{1}{2}\left[\sin(mx+nx) + \sin(mx-nx)\right]$$

$$= \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right]$$

$$\therefore \int \sin mx \cos nx \, dx = \int \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right] dx$$

$$= \frac{1}{2}\left[\frac{-\cos(m+n)x}{(m+n)} + \frac{-\cos(m-n)x}{(m-n)}\right] + c$$

26. $\int \sin mx \sin nx \, dx$

Sol. We have $\sin mx \sin nx = \frac{1}{2}(2\sin mx \sin nx) = \frac{1}{2}\left[\cos(mx-nx) - \cos(mx+nx)\right]$

$$\therefore \int \sin mx \sin nx \, dx = \int \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{(m-n)} - \frac{\sin(m+n)x}{(m+n)}\right] + c$$

27. $\int \cos mx \cos nx \, dx$

Sol. We have, $\cos mx \cos nx = \frac{1}{2}(2\cos mx \cos nx) = \frac{1}{2}\left[\cos(mx+nx) + \cos(mx-nx)\right]$

$$\therefore \int \cos mx \cos nx \, dx = \int \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{(m+n)} + \frac{\sin(m-n)x}{(m-n)}\right] + c$$

28. $\int \frac{\sin x}{\sin(a+x)} dx$

Sol. $\int \frac{\sin x}{\sin(a+x)} dx = \int \frac{\sin((x+a)-a)}{\sin(a+x)} dx$

$$= \int \frac{\sin(x+a)\cos a - \cos(x+a)\sin a}{\sin(a+x)} dx$$

$$= \int \left[\frac{\sin(x+a)\cos a}{\sin(a+x)} - \frac{\cos(x+a)\sin a}{\sin(a+x)}\right] dx$$

$$= \int \left[\cos a - \cot(x+a).\sin a\right] dx$$

$$= \cos a \int 1.dx - \sin a \int \cot(x+a)\, dx$$

$$= (\cos a)(x) - (\sin a)\log|\sin(a+x)| + c$$

29. $\int \frac{1}{7x+3} dx$

Sol. $\int \frac{1}{7x+3} dx = \frac{\log|7x+3|}{7} + c$

30. $\int \frac{\log(1+x)}{1+x} dx$

Sol. $\int \frac{\log(1+x)}{1+x} dx = \int \log(1+x).\frac{1}{1+x} dx = \frac{\left[\log(1+x)\right]^2}{2} + c$

31. $\int \frac{dx}{\sqrt{1+5x}} dx$

Sol. $\int \frac{dx}{\sqrt{1+5x}} dx = \frac{2\sqrt{1+5x}}{5} + c$

32. $\int (1-2x^3)x^2\, dx$

Sol. $\int (1-2x^3)x^2\, dx = \int (x^2 - 2x^5)\, dx = \frac{x^3}{3} - \frac{2x^6}{6} = \frac{x^3}{3} - \frac{x^6}{3} + c$

33. $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$

Sol. Put $1 + \tan x = t \Rightarrow \sec^2 x dx = dt$

$$\int \frac{\sec^2 x}{(1+\tan x)^3} dx = \int \frac{dt}{t^3} = \int t^{-3} dt = \frac{t^{-3+1}}{-3+1} = \frac{t^{-2}}{-2} = \frac{1}{-2t^2} = \frac{1}{-2(1+\tan x)^2} + c$$

34. $\int x^3 \sin x^4\, dx$

Sol. Put $x^4 = t \Rightarrow 4x^3 dx = dt \Rightarrow x^3 dx = \frac{dt}{4}$

$$\int x^3 \sin x^4\, dx = \int (\sin x^4).x^3\, dx$$

$$= \int \sin t \frac{dt}{4} = \frac{1}{4}\int \sin t\, dt = \frac{1}{4}(-\cos t) = \frac{-\cos x^4}{4} + c$$

Sol.

35. $\int \frac{\cos x}{(1+\sin x)^2} dx$

Sol. Put $1+\sin x = t \Rightarrow \cos x\, dx = dt$

$$\int \frac{\cos x}{(1+\sin x)^2} dx = \int \frac{dt}{t^2} dt = \int t^{-2} dt = \frac{t^{-2+1}}{-2+1} = -\frac{1}{t} = -\frac{1}{1+\sin x} + c$$

36. $\int \sqrt[3]{\sin x}\,.\cos x\, dx$

Sol. $$\int \sqrt[3]{\sin x}\,.\cos x\, dx = \int (\sin x)^{\frac{1}{3}}.\cos x\, dx = \frac{(\sin x)^{\frac{1}{3}+1}}{\frac{1}{3}+1} = \frac{3}{4}(\sin x)^{\frac{4}{3}} + c$$

37. $\int 2x e^{x^2}\, dx$

Sol. Put $x^2 = t \Rightarrow 2x\ dx = dt$

$$\int 2x e^{x^2}\, dx = \int e^{x^2}.2x\, dx = \int e^t\, dt = e^t = e^{x^2} + c$$

38. $\int \frac{e^{\log x}}{x} dx$

Sol. Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\int \frac{e^{\log x}}{x} dx = \int e^{\log x} \frac{1}{x} dx = \int e^t\, dt = e^t = e^{\log x} + c$$

39. $\int \frac{x^2}{\sqrt{1-x^6}} dx$

Sol. Put $x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = \frac{dt}{3}$

$$\int \frac{x^2}{\sqrt{1-x^6}} dx = \int \frac{x^2 dx}{\sqrt{1-(x^3)^2}} = \int \frac{1}{\sqrt{1-t^2}} \frac{dt}{3} = \frac{1}{3}\sin^{-1} t = \frac{1}{3}\sin^{-1}(x^3) + c$$

40. $\int \frac{2x^3}{1+x^8} dx$

Sol. Put $x^4 = t \Rightarrow 4x^3 dx = dt \Rightarrow 2.2x^3 dx = dt \Rightarrow 2x^3 dx = \frac{dt}{2}$

$$\int \frac{2x^3}{1+x^8} dx = \int \frac{2x^3 dx}{1+(x^4)^2} = \int \frac{\frac{dt}{2}}{1+t^2} = \frac{1}{2}\int \frac{1}{1+t^2} dt = \frac{1}{2}\tan^{-1} t = \frac{1}{2}\tan^{-1}(x^4) + c$$

41. $\int \frac{x^8}{1+x^{18}} dx$

Sol. Put $x^9 = t \Rightarrow 9x^8 dx = dt \Rightarrow x^8 dx = \frac{dt}{9}$

$$\int \frac{x^8}{1+x^{18}} dx = \int \frac{x^8 dx}{1+(x^9)^2} dx = \int \frac{\frac{dt}{9}}{1+t^2} = \frac{1}{9}\int \frac{1}{1+t^2} dt = \frac{1}{9}\tan^{-1} t = \frac{1}{9}\tan^{-1}(x^9) + c$$

42. $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$

Sol. Put $xe^x = t \Rightarrow (x.e^x + e^x.1)dx = dt \Rightarrow e^x(x+1)dx = dt \Rightarrow e^x(1+x)dx = dt$

$$\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\, dt = \tan t = \tan(xe^x) + c$$

43. $\int \frac{\operatorname{cosec}^2 x}{(a+b\cot x)^5} dx$

Sol. Put $a + b\cot x = t \Rightarrow b(-\operatorname{cosec}^2 x)dx = dt \Rightarrow \operatorname{cosec}^2 x dx = \frac{dt}{-b}$

$$\int \frac{\operatorname{cosec}^2 x}{(a+b\cot x)^5} dx = \int \frac{\frac{dt}{-b}}{t^5} = -\frac{1}{b}\int \frac{1}{t^5} dt$$

$$= -\frac{1}{b}\int t^{-5}\, dt = -\frac{1}{b}\frac{t^{-5+1}}{-5+1} = -\frac{1}{b}\frac{t^{-4}}{-4} = \frac{1}{4bt^4} = \frac{1}{4b\left(a+b\cot x\right)^4} + c$$

44. $\int e^x \sin e^x\, dx$

Sol. Put $e^x = t \Rightarrow e^x dx = dt$

$$\int e^x \sin e^x\, dx = \int \sin t\, dt = -\cos t = -\cos(e^x) + c$$

45. $\int \frac{\sin(\log x)}{x} dx$

Sol. Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\int \frac{\sin(\log x)}{x} dx = \int \sin(\log x).\frac{1}{x} dx = \int \sin t.dt = -\cos t = -\cos(\log x) + c$$

46. $\int \frac{1}{x \log x} dx$

Sol. Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\int \frac{1}{x \log x} dx = \int \frac{1}{\log x} . \frac{1}{x} dx = \int \frac{1}{t} . dt = \log |t| = \log |\log x| + c$$

47. $\int \frac{(1+\log x)^n}{x} dx$

Sol. Put $1 + \log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\int \frac{(1+\log x)^n}{x} dx = \int (1+\log x)^n . \frac{1}{x} dx = \int t^n . dt = \frac{t^{n+1}}{n+1} = \frac{(1+\log x)^{n+1}}{n+1} + c$$

48. $\int \frac{\cos(\log x)}{x} dx$

Sol. Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\int \frac{\cos(\log x)}{x} dx = \int \cos(\log x) . \frac{1}{x} dx = \int \cos t . dt = \sin t = \sin(\log x) + c$$

49. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

Sol. Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\, dt$

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = \int \frac{\cos t}{t} .2t\, dt = 2\int \cos t\, dt = 2 \sin t = 2 \sin \sqrt{x} + c$$

50. $\int \frac{2x+1}{x^2+x+1} dx$

Sol. Put $x^2 + x + 1 = t \Rightarrow (2x+1) dx = dt$

$$\int \frac{2x+1}{x^2+x+1} dx = \int \frac{1}{x^2+x+1} (2x+1)\, dx = \int \frac{1}{t} . dt = \log |t| = \log |x^2+x+1| + c$$

51. $\int \frac{ax^{n-1}}{bx^n + c} dx$

Sol. Put $bx^n + c = t \Rightarrow (b.n.x^{n-1}) dx = dt \Rightarrow x^{n-1} dx = \frac{dt}{bn}$

$$\int \frac{ax^{n-1}}{bx^n + c} dx = \int \frac{1}{bx^n + c} . ax^{n-1}\, dx = \int \frac{1}{t} . a . \frac{dt}{bn}$$

$$= \frac{a}{bn}\int \frac{1}{t}.dt = \frac{a}{bn}\log|t| = \frac{a}{bn}\log|bx^n + c| + c_1$$

52. $\int \frac{1}{x\log x\left[\log(\log x)\right]}dx$

Sol. Put $\log(\log x) = t \Rightarrow \frac{1}{\log x}\frac{d}{dx}(\log x)\,dx = dt$

$$\Rightarrow \frac{1}{\log x}\frac{1}{x}dx = dt \Rightarrow \frac{1}{x\log x}dx = dt$$

$$\int \frac{1}{x\log x\left[\log(\log x)\right]}dx = \int \frac{1}{\log(\log x)}.\frac{1}{x\log x}dx = \int \frac{1}{t}.dt = \log|t| = \log|\log(\log x)| + c$$

53. $\int \coth x\,dx$

Sol. $\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}dx = \log|\sinh x| + c$

54. $\int \frac{1}{(x+3)\sqrt{x+2}}dx$

Sol. Put $\sqrt{x+2} = t \Rightarrow x+2 = t^2 \Rightarrow x = t^2 - 2 \Rightarrow dx = 2t.dt$

$$\int \frac{1}{(x+3)\sqrt{x+2}}dx = \int \frac{1}{(t^2-2+3).t}2t\,dt$$

$$= 2\int \frac{1}{t^2+1}dt = 2\tan^{-1}t = 2\tan^{-1}\sqrt{x+2} + c$$

55. $\int \frac{1}{1+\sin 2x}dx$

Sol. $\int \frac{1}{1+\sin 2x}dx = \int \frac{1}{1+\sin 2x}.\frac{1-\sin 2x}{1-\sin 2x}dx$

$$= \int \frac{1-\sin 2x}{1-\sin^2 2x}dx = \int \frac{1-\sin 2x}{\cos^2 2x}dx$$

$$= \int \left(\frac{1}{\cos^2 2x} - \frac{\sin 2x}{\cos 2x\cos 2x}\right)dx$$

$$= \int \left(\sec^2 2x - \tan 2x\sec 2x\right)dx$$

$$= \frac{\tan 2x}{2} - \frac{\sec 2x}{2} + c$$

56. $\int \frac{x^2+1}{x^4+1}dx$

Sol. $\int \frac{x^2+1}{x^4+1}dx = \int \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}\right)}dx$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+2-2}dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}dx$$

Put $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2+2} = \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{x-\frac{1}{x}}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{x^2-1}{\sqrt{2}x}\right]+c$$

57. $\int \frac{dx}{\cos^2 x+\sin 2x}$

Sol. $\int \frac{dx}{\cos^2 x+\sin 2x} = \int \frac{\sec^2 x}{\sec^2 x(\cos^2 x+\sin 2x)}dx$

$$= \int \frac{\sec^2 x}{\frac{1}{\cos^2 x}(\cos^2 x+\sin 2x)}dx = \int \frac{\sec^2 x}{1+2\tan x}dx$$

Put $1+2\tan x = t \Rightarrow 2\sec^2 x dx = dt \Rightarrow \sec^2 x dx = \frac{dt}{2}$

$$= \int \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int \frac{1}{t}dt = \frac{1}{2}\log|t| = \frac{1}{2}\log|1+2\tan x|+c$$

58. $\int \frac{x^2}{(a+bx)^2}dx$

Sol. Put $a+bx=t \Rightarrow b.dx = dt \Rightarrow dx = \frac{dt}{b},\ x=\frac{t-a}{b} \Rightarrow x^2 = \frac{(t-a)^2}{b^2} = \frac{t^2+a^2-2at}{b^2}$

$$\therefore \int \frac{x^2}{(a+bx)^2}dx = \int \frac{(t^2+a^2-2at)}{b^2t^2}\frac{dt}{b}$$

$$= \frac{1}{b^3}\int\left(\frac{t^2}{t^2}+\frac{a^2}{t^2}-\frac{2at}{t^2}\right)dt = \frac{1}{b^3}\int\left(1+a^2t^{-2}-2a\frac{1}{t}\right)dt$$

$$= \frac{1}{b^3}\left[t+\frac{a^2t^{-1}}{-1}-2a\log|t|\right] = \frac{1}{b^3}\left[t-\frac{a^2}{t}-2a\log|t|\right]$$

$$= \frac{1}{b^3}\left[(a+bx)-\frac{a^2}{a+bx}-2a\log|a+bx|\right]+c$$

59. $\int\sqrt{1+\cos 2x}\,dx$

Sol. $\int\sqrt{1+\cos 2x}\,dx = \int\sqrt{2\cos^2 x}\,dx \quad = \int\sqrt{2}\cos x\,dx = \sqrt{2}\sin x + c$

60. $\int\frac{\cos x+\sin x}{\sqrt{1+\sin 2x}}dx$

Sol. $\int\frac{\cos x+\sin x}{\sqrt{1+\sin 2x}}dx = \int\frac{\cos x+\sin x}{\sqrt{\sin^2 x+\cos^2 x+2\sin x\cos x}}dx$

$$= \int\frac{\cos x+\sin x}{\sqrt{(\sin x+\cos x)^2}}dx = \int\frac{\cos x+\sin x}{\sin x+\cos x}dx = \int 1.dx = x+c$$

61. $\int\frac{\sin 2x}{(a+b\cos x)^2}dx$

Sol. $\int\frac{\sin 2x}{(a+b\cos x)^2}dx = \int\frac{2\sin x\cos x}{(a+b\cos x)^2}dx$

Put $(a+b\cos x)=t \Rightarrow -b\sin x\,dx = dt \Rightarrow \sin x\,dx = \frac{dt}{-b} \Rightarrow \cos x = \frac{t-a}{b}$

$$= \int\frac{2\sin x\cos x}{(a+b\cos x)^2}dx = 2\int\frac{\cos x.\sin x\,dx}{(a+b\cos x)^2}$$

$$= 2\int\frac{t-a}{b}\frac{1}{t^2}\frac{dt}{-b} = \frac{2}{-b^2}\int\left(\frac{t}{t^2}-\frac{a}{t^2}\right)dt$$

$$= \frac{2}{-b^2}\int\left(\frac{1}{t}-at^{-2}\right)dt = \frac{2}{-b^2}\left[\log|t|-\frac{at^{-2+1}}{-2+1}\right]$$

$$= \frac{2}{-b^2}\left[\log|t|+\frac{a}{t}\right] = \frac{2}{-b^2}\left[\log|a+b\cos x|+\frac{a}{a+b\cos x}\right]+c$$

62. $\int \frac{\sec x}{(\sec x+\tan x)^2}dx$

Sol. Put $(\sec x+\tan x)=t \Rightarrow (\sec x\tan x+\sec^2 x)\,dx=dt$

$$\Rightarrow \sec x(\tan x+\sec x)\,dx=dt \Rightarrow \sec x(t)dx=dt \Rightarrow \sec x\,dx=\frac{dt}{t}$$

$$\therefore \int \frac{\sec x}{(\sec x+\tan x)^2}dx=\int \frac{1}{t^2}\frac{dt}{t}=\int \frac{dt}{t^3}$$

$$=\int t^{-3}dt=\frac{t^{-3+1}}{-3+1}=-\frac{1}{2t^2}=-\frac{1}{2(\sec x+\tan x)^2}+c$$

63. $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$

Sol. $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}=\int \frac{\sec^2 x\,dx}{\sec^2 x(a^2\sin^2 x+b^2\cos^2 x)}$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$$=\int \frac{\sec^2 x\,dx}{a^2\tan^2 x+b^2}=\int \frac{dt}{a^2t^2+b^2}=\int \frac{1}{(at)^2+b^2}dt$$

$$=\frac{1}{b}\frac{\tan^{-1}\left(\frac{at}{b}\right)}{a}=\frac{1}{ab}\tan^{-1}\left(\frac{a(\tan x)}{b}\right)+c$$

64. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

Sol. $\int \frac{dx}{\sin(x-a)\sin(x-b)}=\int \frac{1}{\sin(b-a)}\frac{\sin(b-a)}{\sin(x-a)\sin(x-b)}dx$

$$=\int \frac{\sin\left[(x-a)-(x-b)\right]}{\sin(b-a)\sin(x-a)\sin(x-b)}dx \qquad \left[\because b-a=(x-a)-(x-b)\right]$$

$$=\frac{1}{\sin(b-a)}\int \frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}dx$$

$$=\frac{1}{\sin(b-a)}\int \left[\frac{\sin(x-a)\cos(x-b)}{\sin(x-a)\sin(x-b)}-\frac{\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}\right]dx$$

$$=\frac{1}{\sin(b-a)}\int \left[\cot(x-b)-\cot(x-a)\right]dx$$

$$=\frac{1}{\sin(b-a)}\left[\frac{\log|\sin(x-b)|}{1}-\frac{\log|\sin(x-a)|}{1}\right] =\frac{1}{\sin(b-a)}\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+c$$

65. $\int \frac{dx}{\cos(x-a)\cos(x-b)}$

Sol. $\int \frac{dx}{\cos(x-a)\cos(x-b)} = \int \frac{1}{\sin(b-a)} \frac{\sin(b-a)}{\cos(x-a)\cos(x-b)} dx$

$$= \frac{1}{\sin(b-a)} \int \frac{\sin[(x-a)-(x-b)]}{\cos(x-a)\cos(x-b)} dx$$

$$= \frac{1}{\sin(b-a)} \int \left[\frac{\sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)} \right] dx$$

$$= \frac{1}{\sin(b-a)} \int \left[\frac{\sin(x-a)\cos(x-b)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)} \right] dx$$

$$= \frac{1}{\sin(b-a)} \int [\tan(x-b) - \tan(x-a)] dx$$

$$= \frac{1}{\sin(b-a)} [\log|\sec(x-a)| - \log|\sec(x-b)|] + c$$

$$= \frac{1}{\sin(b-a)} \left[\log\left| \frac{\sec(x-a)}{\sec(x-b)} \right| \right] + c$$

66. $\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x} dx$

Sol. $\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x} dx = \int \frac{2\sin x\cos x}{a\cos^2 x + b\sin^2 x} dx$

Put $(a\cos^2 x + b\sin^2 x) = t \Rightarrow [a2\cos x(-\sin x) + b2\sin x\cos x] dx = dt$

$$2\sin x\cos x(-a+b)dx = dt \Rightarrow 2\sin x\cos x\, dx = \frac{dt}{b-a}$$

$$\int \frac{2\sin x\cos x}{a\cos^2 x + b\sin^2 x} dx = \int \frac{1}{t}\frac{dt}{b-a}$$

$$= \frac{1}{b-a}\int \frac{1}{t} dt = \frac{1}{b-a}\log|t| = \frac{1}{b-a}\log|a\cos^2 x + b\sin^2 x| + c$$

67. $\int \frac{1-\tan x}{1+\tan x} dx$

Sol. $\int \frac{1-\tan x}{1+\tan x} dx = \int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

Put $(\cos x+\sin x)=t \Rightarrow (-\sin x+\cos x)\,dx=dt$

$$=\int \frac{dt}{t}=\log|t|=\log|\sin x+\cos x|+c$$

68. $\displaystyle\int \frac{\cot(\log x)}{x}dx$

Sol. Put $\log x=t \Rightarrow \frac{1}{x}dx=dt$

$$\int \frac{\cot(\log x)}{x}dx=\int \cot(\log x).\frac{1}{x}.dx$$

$$=\int \cot t.dt=\log|\sin t|=\log|\sin(\log x)|+c$$

69. $\displaystyle\int e^x.\cot e^x.dx$

Sol. Put $e^x=t \Rightarrow e^x dx=dt$

$$\int e^x.\cot e^x.dx=\int \cot e^x.e^x.dx$$

$$=\int \cot t.dt=\log|\sin t|=\log|\sin e^x|+c$$

70. $\displaystyle\int \frac{2x+3}{\sqrt{x^2+3x-4}}dx$

Sol. $\displaystyle\int \frac{2x+3}{\sqrt{x^2+3x-4}}dx=2\sqrt{x^2+3x-4}+c \qquad \left(\because \int \frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}\right)$

71. $\displaystyle\int \mathrm{cosec}^2 x\sqrt{\cot x}\,dx$

Sol. Put $\cot x=t \Rightarrow -\mathrm{cosec}^2 x\,dx=dt \Rightarrow \mathrm{cosec}^2 x dx=-dt$

$$\int \mathrm{cosec}^2 x\sqrt{\cot x}\,dx=\int \sqrt{\cot x}\,\mathrm{cosec}^2 x.dx$$

$$=\int \sqrt{t}(-dt)=-\int t^{1/2}\,dt=\frac{-t^{3/2}}{\frac{3}{2}}=-\frac{2}{3}(\cot x)^{3/2}+c$$

72. $\displaystyle\int \sec x\log(\sec x+\tan x)\,dx$

Sol. Put $\log(\sec x+\tan x)=t$

$$\Rightarrow \frac{1}{\sec x+\tan x}\left[\sec x\tan x+\sec^2 x\right]dx=dt$$

$$\Rightarrow \frac{\sec x(\tan x+\sec x)}{(\sec x+\tan x)}dx=dt \Rightarrow \sec x\,dx=dt$$

$$\int \sec x \log(\sec x + \tan x)\,dx = \int \log(\sec x + \tan x).\sec x.dx$$

$$= \int t.dt = \frac{t^2}{2} = \frac{[\log(\sec x + \tan x)]^2}{2} + c$$

73. $\int \cos^3 x\,dx$

Sol. Put $\cos 3x = 4\cos^3 x - 3\cos x \Rightarrow \cos^3 x = \dfrac{\cos 3x + 3\cos x}{4}$

$$\int \cos^3 x\,dx = \int \frac{\cos 3x + 3\cos x}{4}dx$$

$$= \frac{1}{4}\int (\cos 3x + 3\cos x)\,dx = \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right] = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + c$$

74. $\int x\sqrt{4x+3}\,dx$

Sol. Put $\sqrt{4x+3} = t \Rightarrow 4x+3 = t^2 \Rightarrow 4.d\,x = 2t\,dt \Rightarrow dx = \dfrac{1}{2}t\,dt \Rightarrow x = \dfrac{t^2-3}{4}$

$$\int x\sqrt{4x+3}\,dx = \int \frac{t^2-3}{4}.t.\frac{t}{2}.dt = \frac{1}{8}\int (t^2-3).t^2.dt$$

$$= \frac{1}{8}\int (t^4 - 3t^2)\,dt = \frac{1}{8}\left[\frac{t^5}{5} - \frac{3t^3}{3}\right] = \frac{t^5}{40} - \frac{t^3}{8}$$

$$= \frac{(\sqrt{4x+3})^5}{40} - \frac{(\sqrt{4x+3})^3}{8} = \frac{(4x+3)^{5/2}}{40} - \frac{(4x+3)^{3/2}}{8} + c$$

75. $\int \dfrac{1}{a^2 + (b+cx)^2}dx$

Sol. $\int \dfrac{1}{a^2+(b+cx)^2}dx = \dfrac{1}{a}\dfrac{\tan^{-1}\left(\dfrac{b+cx}{a}\right)}{c}$ $\quad\left(\because \int \dfrac{1}{a^2+x^2}dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)\right)$

$$= \frac{1}{ac}\tan^{-1}\left(\frac{b+cx}{a}\right) + c$$

Unit **7**

Definite Integrals

The fundamental theorem of Integral Calculus

If f is integrable on [a, b] and if there is a differentiable function F on [a, b] such that F' = f, then $\int_a^b f(x) = \mathrm{F(b)} - \mathrm{F(a)}$. We call $\int_a^b f(x)\,dx$, the definite integral of f from, a to b. 'a' is called the lower limit, 'b' is called the upper limit of the integral.

The letter 'x' is called the variable of integration.

Note: We write $\left[\mathrm{F}(x)\right]_a^b$ for F(b) – F(a). Also $\left[\mathrm{F}(x)\right]_a^b$ is not dependent on x and $\left[\mathrm{F}(x)\right]_a^b = -\left[\mathrm{F}(x)\right]_b^a$.

The function f in $\int_a^b f(x)\,dx$ is called the 'integrand'. The numerical value of $\int_a^b f(x)\,dx$ depends on f and does not dependent on the symbol x. The letter 'x' is a "dummy symbol" and may be replaced by any other convenient symbol.

Properties:

1. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

2. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where a < c < b.

5. $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \\ 0, & \text{if } f(-x) = -f(x) \end{cases}$$

6. $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f\left(x\right)dx. & \text{if} \quad f(2a-x)=f(x) \\ 0, & \text{if} \quad f(2a-x)=-f(x) \end{cases}$$

Problems

1. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x+\cos^{\frac{5}{2}} x}dx$.

Sol. Let $I = \int_0^{\pi/2} \frac{\cos^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x+\cos^{\frac{5}{2}} x}dx$ ....(1)

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^{\frac{5}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{5}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{5}{2}}\left(\frac{\pi}{2}-x\right)}dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f\left(a-x\right)dx\right]$$

$$= \int_0^{\pi/2} \frac{\sin^{\frac{5}{2}} x}{\cos^{\frac{5}{2}} x+\sin^{\frac{5}{2}} x}dx \qquad ....(2)$$

Adding (1) and (2), we get

$$I+I = \int_0^{\pi/2} \frac{\cos^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x+\cos^{\frac{5}{2}} x}dx + \int_0^{\pi/2} \frac{\sin^{\frac{5}{2}} x}{\cos^{\frac{5}{2}} x+\sin^{\frac{5}{2}} x}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left(\frac{\cos^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x+\cos^{\frac{5}{2}} x}+\frac{\sin^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x+\cos^{\frac{5}{2}} x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{5}{2}} x+\sin^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x+\cos^{\frac{5}{2}} x}dx$$

$$= \int_0^{\frac{\pi}{2}} 1.dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}-0=\frac{\pi}{2}$$

$$\Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

$$\therefore\ \mathrm{I} = \int_0^{\pi/2} \frac{\cos^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x + \cos^{\frac{5}{2}} x}\,dx = \frac{\pi}{4}$$

2. Show that $\displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$.

Sol. Let $\mathrm{I} = \displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$

But $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where $a = \dfrac{\pi}{2}$ here

$$\therefore\ \mathrm{I} = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)}{\cos x + \sin x}\,dx$$

$$= \int_0^{\pi/2} \left(\frac{\frac{\pi}{2}}{\sin x + \cos x} - \frac{x}{\sin x + \cos x} \right) dx$$

$$= \int_0^{\pi/2} \frac{\frac{\pi}{2}}{\sin x + \cos x}\,dx - \int_0^{\pi/2} \frac{x}{\sin x + \cos x}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}\,dx - \mathrm{I}$$

$$\Rightarrow \mathrm{I} + \mathrm{I} = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}\,dx$$

$$\Rightarrow \mathrm{I} = \frac{\pi}{4}\int_0^{\pi/2} \frac{1}{\sin x + \cos x}\,dx$$

Put $t = \tan\dfrac{x}{2} \Rightarrow dt = \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx,\ \sin x = \dfrac{2t}{1+t^2},\ \cos x = \dfrac{1-t^2}{1+t^2}$ and $\sec^2\dfrac{x}{2} = 1+t^2$

When $x = 0, t = 0$ and when $x = \dfrac{\pi}{2}, t = 1$. Thus

$$\text{I} = \frac{\pi}{4}\int_0^{\pi/2} \frac{\frac{1}{2}\sec^2\frac{x}{2}}{(\sin x+\cos x)\left(\frac{1}{2}\sec^2\frac{x}{2}\right)}dx = \frac{\pi}{4}\int_0^1 \frac{2dt}{2t+1-t^2}$$

$$= \frac{\pi}{4}\int_0^1 \frac{dt}{(\sqrt{2})^2-(t-1)^2}$$

$$= \frac{\pi}{4}\left[\frac{1}{2\sqrt{2}}\log\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right]_0^1 = \frac{-\pi}{4\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = \frac{-\pi}{2\sqrt{2}}\cdot\frac{1}{2}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$= \frac{\pi}{2\sqrt{2}}\log\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}} = \frac{\pi}{2\sqrt{2}}\log\left(\sqrt{2}+1\right).$$

3. Show that $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$.

Sol. Let $\text{I} = \int_0^{\pi/2} \sin^n x\, dx$

Let $a = \frac{\pi}{2}$, $f(x) = \sin^n x = (\sin x)^n$

$$f(a-x) = f\left(\frac{\pi}{2}-x\right) = \left[\sin\left(\frac{\pi}{2}-x\right)\right]^n$$

$$= (\cos x)^n = \cos^n x$$

We know that, $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$\Rightarrow \int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$ Hence proved.

4. Evalute $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx.$

Sol. Let $a = \frac{\pi}{6}$, $b = \frac{\pi}{3}$, $f(x) = \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}$

Then, $a+b-x = \frac{\pi}{6}+\frac{\pi}{3}-x = \left(\frac{\pi}{2}-x\right)$

$$\therefore f(a+b-x) = f\left(\frac{\pi}{2}-x\right) = \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}} = \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}$$

Let $\mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx.$

We know that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$\therefore \mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \Rightarrow \mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx.$$

Adding them, we get

$$\mathrm{I}+\mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

$$2\mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} \right) dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

$$2\,\mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1.dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

$$\mathrm{I} = \frac{\pi}{6\times 2} = \frac{\pi}{12} \Rightarrow \mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx = \frac{\pi}{12}.$$

5. Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$.

Sol. Let $I = \int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$

Let $a = \pi, f(x) = \frac{x\sin x}{1+\sin x}$

Then $f(a-x) = f(\pi - x) = \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)}$

$$= \frac{(\pi - x)\sin x}{1+\sin x}$$

We know that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore\ I = \int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x}\,dx$$

$$\therefore\ I = \int_0^{\pi} \frac{\pi\sin x - x\sin x}{1+\sin x}\,dx$$

$$= \int_0^{\pi}\left(\frac{\pi\sin x}{1+\sin x} - \frac{x\sin x}{1+\sin x}\right)dx$$

$$= \int_0^{\pi} \frac{\pi\sin x}{1+\sin x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$$

$$I = \pi\int_0^{\pi} \frac{\sin x}{1+\sin x}\,dx - I$$

$$\therefore\ I + I = \pi\int_0^{\pi} \frac{\sin x}{1+\sin x}\,dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \frac{1+\sin x - 1}{1+\sin x}\,dx$$

$$\Rightarrow I = \frac{\pi}{2}\int_0^{\pi}\left(\frac{1+\sin x}{1+\sin x} - \frac{1}{1+\sin x}\right)dx$$

$$= \frac{\pi}{2}\int_0^{\pi}\left(1 - \frac{1}{1+\sin x}\right)dx$$

$$= \frac{\pi}{2}\left[\int_0^{\pi} 1.dx - \int_0^{\pi} \frac{1}{1+\sin x}dx\right]$$

$$= \frac{\pi}{2}\left[(x)_0^{\pi} - \int_0^{\pi} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}dx\right]$$

$$= \frac{\pi}{2}\left[\pi - \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}dx\right]$$

$$= \frac{\pi^2}{2} - \frac{\pi}{2}\int_0^{\pi}\left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)dx$$

$$= \frac{\pi^2}{2} - \frac{\pi}{2}\int_0^{\pi}\left(\sec^2 x - \sec x \tan x\right)dx$$

$$= \frac{\pi^2}{2} - \frac{\pi}{2}\left(\tan x - \sec x\right)_0^{\pi}$$

$$= \frac{\pi^2}{2} - \frac{\pi}{2}\left[\left(\tan \pi - \sec \pi\right) - \left(\tan 0 - \sec 0\right)\right]$$

$$= \frac{\pi^2}{2} - \frac{\pi}{2}\left[0 - (-1) - 0 + 1\right]$$

$$\mathrm{I} = \frac{\pi^2}{2} - \frac{\pi}{2}(2) = \frac{\pi^2}{2} - \pi$$

$$\therefore \int_0^{\pi} \frac{x \sin x}{1+\sin x}dx = \frac{\pi^2}{2} - \pi$$

6. Evaluate $\int_1^4 x\sqrt{x^2-1}\,dx$.

Sol. $\int_1^4 x\sqrt{x^2-1}\,dx = \int_1^4 \left(x^2-1\right)^{1/2}.x\,dx$

$$= \frac{1}{2}\int_1^4 \left(x^2-1\right)^{1/2}.2x\,dx \qquad \left[\because \int \left[f(x)\right]^n.f'(x)dx = \frac{\left[f(x)\right]^{n+1}}{n+1}\right]$$

$$= \frac{1}{2}\left[\frac{\left(x^2-1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4$$

$$= \left[\frac{1}{2} \frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4$$

$$= \frac{1}{2} \times \frac{2}{3} \left[(4^2-1)^{3/2} - (1-1)^{3/2} \right] = \frac{1}{3}\left[(15)^{3/2} \right]$$

7. Evaluate $\int_0^2 \sqrt{4-x^2}\, dx$.

Sol. $\int_0^2 \sqrt{4-x^2}\, dx = \int_0^2 \sqrt{2^2-x^2}\, dx$ $\left(\because \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right)$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right]_0^2$$

$$= \left[\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}\left(\frac{2}{2}\right) \right] - \left[\frac{0}{2}\sqrt{4-0} + 2\sin^{-1} 0 \right]$$

$$= 0 + 2\sin^{-1}(1) - 0 - 0$$

$$= 2 \times \frac{\pi}{2} = \pi.$$

8. Evalute $\int_{-\pi/2}^{\pi/2} \sin|x|\, dx$.

Sol. $\therefore \int_{-\pi/2}^{\pi/2} \sin|x|\, dx = \int_{-\pi/2}^{0} \sin|x|\, dx + \int_0^{\pi/2} \sin|x|\, dx$

$$= \int_{-\pi/2}^{0} \sin(-x)\, dx + \int_0^{\pi/2} \sin x dx \qquad \left(\because -\frac{\pi}{2} < x < 0 \Rightarrow |x| = -x,\ 0 < x < \frac{\pi}{2} \Rightarrow |x| = x \right)$$

$$= \int_{-\pi/2}^{0} -\sin x\, dx + (-\cos x)_0^{\pi/2}$$

$$= [\cos x]_{-\pi/2}^{0} + \left(-\cos\frac{\pi}{2} - (-\cos 0) \right)$$

$$= \left[\cos 0 - \cos\left(-\frac{\pi}{2}\right) \right] + (-0+1)$$

$$= 1 - 0 - 0 + 1 = 2.$$

9. Evaluate $\int_2^3 \frac{2x}{1+x^2}\,dx.$

Sol. $\int_2^3 \frac{2x}{1+x^2}\,dx = \left[\log\left|1+x^2\right|\right]_2^3$ $\left(\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)|\right)$

$$= \log 10 - \log 5$$

$$= \log\left(\frac{10}{5}\right) = \log 2.$$

10. Evaluate $\int_0^{\pi} \sqrt{2+2\cos\theta}\,d\theta.$

Sol. $\int_0^{\pi} \sqrt{2(1+\cos\theta)}\,d\theta = \int_0^{\pi} \sqrt{4.\cos^2\frac{\theta}{2}}\,d\theta$

$$= \int_0^{\pi} 2.\cos\left(\frac{\theta}{2}\right)d\theta = \left(2.\frac{\sin\left(\frac{\theta}{2}\right)}{\frac{1}{2}}\right)_0^{\pi}$$

$$= \left[4\sin\frac{\theta}{2}\right]_0^{\pi} = \left[4\sin\frac{\pi}{2} - 4\sin 0\right] = 4.$$

11. Evaluate $\int_0^{\pi} \sin^3 x\cos^3 x\,dx.$

Sol. Let $I = \int_0^{\pi} \sin^3 x\cos^3 x\,dx$

We have $\int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx$

$$\Rightarrow I = \int_0^{\pi} \sin^3 x\cos^3 x\,dx = \int_0^{\pi} \sin^3(\pi - x)\cos^3(\pi - x)\,dx$$

$$= -\int_0^{\pi} \sin^3 x\cos^3 x\,dx = -I$$

$\therefore\ I = -I \Rightarrow 2I = 0 \Rightarrow I = 0.$

$\therefore \int_0^{\pi} \sin^3 x\cos^3 x\,dx = 0.$

12. Evaluate $\int_0^2 |1-x|\,dx.$

Sol. $\int_0^2 |1-x|\,dx = \int_0^1 |1-x|\,dx + \int_1^2 |1-x|\,dx$

$= \int_0^1 (1-x)dx + \int_1^2 (-1+x)\,dx \qquad (\because 0<x<1 \Rightarrow |1-x| = +(1-x),\ 1<x<2 \Rightarrow |1-x| = -(1-x) = (-1+x))$

$= \left(x - \frac{x^2}{2}\right)_0^1 + \left(-x + \frac{x^2}{2}\right)_1^2$

$= \left[\left(1-\frac{1}{2}\right) - (0-0)\right] + \left[\left(-2+\frac{4}{2}\right) - \left(-1+\frac{1}{2}\right)\right]$

$= 1 - \frac{1}{2} - 2 + 2 + 1 - \frac{1}{2} = 1.$

13. Evaluate $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\,dx.$

Sol. Let $\mathrm{I} = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\,dx$

Let $a = -\frac{\pi}{2},\ b = \frac{\pi}{2},\ f(x) = \frac{\cos x}{1+e^x}$

$f(a+b-x) = f\left(-\frac{\pi}{2} + \frac{\pi}{2} - x\right) = f(-x) = \frac{\cos(-x)}{1+e^{-x}}$

$= \frac{\cos x}{1+\frac{1}{e^x}} = \frac{\cos x}{e^x+1} \times e^x$

$= \frac{e^x.\cos x}{1+e^x}$

We know that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$\therefore\ \mathrm{I} = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}\,dx \Rightarrow \mathrm{I} = \int_{-\pi/2}^{\pi/2} \frac{e^x.\cos x}{1+e^x}\,dx$

Adding them, we get

$$\mathrm{I}+\mathrm{I}=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^{x}}dx+\int_{-\pi/2}^{\pi/2}\frac{e^{x}.\cos x}{1+e^{x}}dx$$

$$\Rightarrow 2\mathrm{I}=\int_{-\pi/2}^{\pi/2}\left(\frac{\cos x}{1+e^{x}}+\frac{e^{x}.\cos x}{1+e^{x}}\right)dx=\int_{-\pi/2}^{\pi/2}\frac{\cos x+e^{x}.\cos x}{1+e^{x}}dx$$

$$\Rightarrow 2\mathrm{I}=\int_{-\pi/2}^{\pi/2}\frac{\cos x(1+e^{x})}{1+e^{x}}dx=\int_{-\pi/2}^{\pi/2}\cos x\,dx$$

$$\Rightarrow 2\mathrm{I}=\left(\sin x\right)_{-\pi/2}^{\pi/2}=\sin\frac{\pi}{2}-\sin\left(-\frac{\pi}{2}\right)$$

$\Rightarrow 2\mathrm{I} = 1 - (-1) = 2$

$$\Rightarrow \mathrm{I}=1\Rightarrow \mathrm{I}=\int_{-\pi/2}^{\pi/2}\frac{\cos x}{1+e^{x}}dx=1.$$

14. Evaluate $\displaystyle\int_{0}^{3}\frac{x}{\sqrt{x^{2}+16}}dx$.

Sol. $$\int_{0}^{3}\frac{x}{\sqrt{x^{2}+16}}dx=\frac{1}{2}\int_{0}^{3}\frac{2x}{\sqrt{x^{2}+16}}dx$$

$$=\left(\frac{1}{2}.2\sqrt{x^{2}+16}\right)_{0}^{3}$$

$$=\left(\sqrt{x^{2}+16}\right)_{0}^{3}$$

$$=\sqrt{3^{2}+16}-\sqrt{0^{2}+16}$$

$= 5 - 4$

$= 1.$

15. Evaluate $\displaystyle\int_{0}^{1}x.e^{-x^{2}}\,dx$.

Sol. $$\int_{0}^{1}x.e^{-x^{2}}\,dx=\int_{0}^{1}e^{-x^{2}}.x\,dx$$

Put $-x^2 = \mathrm{t} \Rightarrow -2x\,dx = dt \Rightarrow x\,dx=\dfrac{-dt}{2}$

Upper Limit : $x=1\Rightarrow \mathrm{t}=-1$ and Lower Limit : $x=0\Rightarrow \mathrm{t}=0$

$$\int_0^1 x.e^{-x^2}\,dx = \int_0^{-1} e^t.\left(\frac{-dt}{2}\right)$$

$$= -\frac{1}{2}\int_0^{-1} e^t dt = -\frac{1}{2}\left(e^t\right)_0^{-1}$$

$$= -\frac{1}{2}\left(e^{-1} - e^0\right) = -\frac{1}{2}\left(\frac{1}{e} - 1\right)$$

$$= -\frac{1}{2e} + \frac{1}{2} = \left(\frac{1}{2} - \frac{1}{2e}\right).$$

16. Evaluate $\int_1^5 \frac{1}{\sqrt{2x-1}}\,dx$.

Sol. $$\int_1^5 \frac{dx}{\sqrt{2x-1}}\,dx = \left(\frac{2\sqrt{2x-1}}{2}\right)_1^5$$

$$= \left(\sqrt{2x-1}\right)_1^5$$

$$= \sqrt{10-1} - \sqrt{2-1} = \sqrt{9} - \sqrt{1}$$

$$= 3 - 1 = 2.$$

17. Evaluate $\int_0^4 \frac{x^2}{1+x}\,dx$.

Sol. $$\int_0^4 \frac{x^2}{1+x}\,dx = \int_0^4 \left[(x-1) + \frac{1}{x+1}\right]dx$$

$$= \left[\frac{x^2}{2} - x + \log|x+1|\right]_0^4$$

$$= \left(\frac{4^2}{2} - 4 + \log|4+1|\right) - \left(\frac{0}{2} - 0 + \log 1\right)$$

$$= 8 - 4 + \log 5 - 0$$

$$= (4 + \log 5)$$

18. Evaluate $\int_{-1}^2 \frac{x^2}{x^2+2}\,dx$.

Sol. $$\int_{-1}^2 \frac{x^2}{x^2+2}\,dx = \int_{-1}^2 \left(1 + \frac{-2}{x^2+2}\right)dx$$

$$= \int_{-1}^{2} 1\, dx - 2\int_{-1}^{2} \frac{1}{x^2 + \left(\sqrt{2}\right)^2}\, dx$$

$$= [x]_{-1}^{2} - \left[2.\frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}}\right]_{-1}^{2}$$

$$= [2 - (-1)] - \frac{2}{\sqrt{2}}\left(\tan^{-1} \frac{2}{\sqrt{2}} - \tan^{-1}\left(\frac{-1}{\sqrt{2}}\right)\right)$$

$$= 3 - \sqrt{2}\left(\tan^{-1} \sqrt{2} + \tan^{-1} \frac{1}{\sqrt{2}}\right)$$

19. Evaluate $\int_{0}^{4} |2 - x|\, dx$.

Sol. $\int_{0}^{4} |2 - x|\, dx = \int_{0}^{2} |2 - x|\, dx + \int_{2}^{4} |2 - x|\, dx$

$$= \int_{0}^{2} (2 - x)\, dx + \int_{2}^{4} (-2 + x)\, dx$$

$$= \left[2x - \frac{x^2}{2}\right]_{0}^{2} + \left[-2x + \frac{x^2}{2}\right]_{2}^{4}$$

$$= \left[4 - \frac{4}{2} - 0\right] + \left[\left(-8 + \frac{16}{2}\right) - \left(-4 + \frac{4}{2}\right)\right]$$

$= 2 + [0 + 4 – 2]$

$= 4$

20. Evaluate $\int_{0}^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x}$.

Sol. Let $I = \int_{0}^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x}$ ....(1)

We know that $\int_{0}^{a} f(x)\, dx = \int_{0}^{a} f(a - x)\, dx$

$$I = \int_{0}^{\pi/2} \frac{\sin^5 x}{\sin^5 x + \cos^5 x} = \int_{0}^{\pi/2} \frac{\sin^5\left(\frac{\pi}{2} - x\right)}{\sin^5\left(\frac{\pi}{2} - x\right) + \cos^5\left(\frac{\pi}{2} - x\right)} dx = \int_{0}^{\pi/2} \frac{\cos^5 x}{\cos^5 x + \sin^5 x} dx \quad \text{........(2)}$$

Adding (1) and (2),

$$\therefore\ I+I=\int_0^{\pi/2}\frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx+\int_0^{\pi/2}\frac{\cos^5 x}{\cos^5 x+\sin^5 x}dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}\left(\frac{\sin^5 x}{\sin^5 x+\cos^5 x}+\frac{\cos^5 x}{\cos^5 x+\sin^5 x}\right)dx$$

$$=\int_0^{\pi/2}\left(\frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x}\right)dx$$

$$=\int_0^{\pi/2}1.dx=[x]_0^{\pi/2}=\frac{\pi}{2}$$

$$\Rightarrow 2I=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

$$\therefore I=\int_0^{\pi/2}\frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx=\frac{\pi}{4}$$

21. Evaluate $\int_0^{\pi/2}\frac{\sin^2 x-\cos^2 x}{\sin^3 x+\cos^3 x}dx$.

Sol. Let $I=\int_0^{\pi/2}\frac{\sin^2 x-\cos^2 x}{\sin^3 x+\cos^3 x}dx$ ....(1)

We know that $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$$\therefore\ I=\int_0^{\pi/2}\frac{\sin^2 x-\cos^2 x}{\sin^3 x+\cos^3 x}dx\Rightarrow I=\int_0^{\pi/2}\frac{\sin^2\left(\frac{\pi}{2}-x\right)-\cos^2\left(\frac{\pi}{2}-x\right)}{\sin^3\left(\frac{\pi}{2}-x\right)+\cos^3\left(\frac{\pi}{2}-x\right)}dx$$

$$I=\int_0^{\pi/2}\frac{\cos^2 x-\sin^2 x}{\cos^3 x+\sin^3 x}dx \qquad ....(2)$$

Adding (1) and (2),

$$\therefore\ I+I=\int_0^{\pi/2}\frac{\sin^2 x-\cos^2 x}{\sin^3 x+\cos^3 x}dx+\int_0^{\pi/2}\frac{\cos^2 x-\sin^2 x}{\cos^3 x+\sin^3 x}dx$$

$$2\,\mathrm{I} = \int_0^{\pi/2} \left(\frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} + \frac{\cos^2 x - \sin^2 x}{\cos^3 x + \sin^3 x} \right) dx$$

$$= \int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x + \cos^2 x - \sin^2 x}{\sin^3 x + \cos^3 x} dx$$

$$2\,\mathrm{I} = \int_0^{\pi/2} 0 \; dx = 0$$

$$\Rightarrow \mathrm{I} = 0$$

$$\therefore \int_0^{\pi/2} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx = 0.$$

22. Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{4 + 5\cos x}$.

Sol. Let $\tan\dfrac{x}{2} = t \Rightarrow dx = \dfrac{2dt}{1+t^2} \Rightarrow \cos x = \dfrac{1-t^2}{1+t^2}$

Upper Limit : $x = 0 \Rightarrow \mathrm{t} = \tan 0 = 0$ and Lower Limit : $x = \dfrac{\pi}{2} \Rightarrow t = \tan\dfrac{\pi}{4} = 1$

$$\Rightarrow \int_0^{\pi/2} \frac{dx}{4 + 5\cos x} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{4 + 5\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_0^1 \frac{2dt}{4(1+t^2) + 5\left(1-t^2\right)}$$

$$= 2\int_0^1 \frac{2dt}{9 - t^2} = 2\int_0^1 \frac{1}{3^2 - t^2} dt$$

$$= 2.\frac{1}{2(3)} \left[\log\left|\frac{3+t}{3-t}\right| \right]_0^1$$

$$= \frac{1}{3}\left[\log\left(\frac{4}{2}\right) - \log(1) \right] = \frac{1}{3}\log 2$$

23. Evaluate $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$.

Sol. $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)\, dx = dt$

$t^2 = (\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x$

$t^2 = 1 - \sin 2x$

$\Rightarrow \sin 2x = -t^2 + 1 = 1 - t^2$

U.L.: $x = \frac{\pi}{4} \Rightarrow t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$

L.L.: $x = 0 \Rightarrow t = \sin 0 - \cos 0 = -1$

$$\therefore \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx = \int_{-1}^{0} \frac{dt}{9 + 16(-t^2 + 1)}$$

$$= \int_{-1}^{0} \frac{dt}{-16t^2 + 25}$$

$$= \int_{-1}^{0} \frac{dt}{5^2 - (4t)^2}$$

$$= \left[\frac{\frac{1}{2(5)} . \log\left|\frac{5+4t}{5-4t}\right|}{4} \right]_{-1}^{0}$$

$$= \frac{1}{40}\left[\log 1 - \log\left|\frac{5+4(-1)}{5-4(-1)}\right| \right]$$

$$= \frac{1}{40}\left(0 - \log\frac{1}{9}\right)$$

$$= \frac{-1}{40} . \log\frac{1}{9}$$

$$= -\frac{1}{40} . \log 9^{-1} = \frac{1}{40}\log 9 = \frac{1}{40}\log 3^2$$

$$= \frac{1}{20}\log 3.$$

24. Evaluate $\int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx$.

Sol. Let $I = \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx$ .... (1)

We know that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore\ I = \int_0^{\frac{\pi}{2}} \frac{a\sin\left(\frac{\pi}{2}-x\right) + b\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{a\cos x + b\sin x}{\cos x + \sin x}\,dx \qquad \text{........ (2)}$$

Adding (1) and (2), we get

$$I + I = \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx + \int_0^{\frac{\pi}{2}} \frac{a\cos x + b\sin x}{\cos x + \sin x}\,dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x + a\cos x + b\sin x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{a(\sin x + \cos x) + b(\cos x + \sin x)}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(a+b)(\sin x + \cos x)}{\sin x + \cos x}\,dx$$

$$= (a+b)\int_0^{\frac{\pi}{2}} 1.dx = (a+b)(x)_0^{\pi/2}$$

$$2\,I = (a+b)\frac{\pi}{2}$$

$$\therefore\ I = (a+b)\frac{\pi}{4}$$

25. Evaluate $\int_0^{\pi} \frac{x}{1+\sin x} dx$.

Sol. Let $I = \int_0^{\pi} \frac{x}{1+\sin x} dx$

We know that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)} dx = \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx$$

$$\Rightarrow I = \int_0^{\pi} \left(\frac{\pi}{1+\sin x} - \frac{x}{1+\sin x} \right) dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\sin x} dx - \int_0^{\pi} \frac{x}{1+\sin x} dx$$

$$= \pi \int_0^{\pi} \frac{1}{1+\sin x} dx - I$$

$$\therefore I + I = \pi \int_0^{\pi} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \frac{\pi}{2} \int_0^{\pi} (\sec^2 x - \sec x \tan x)\, dx$$

$$= \frac{\pi}{2} \left[\tan x - \sec x \right]_0^{\pi}$$

$$= \frac{\pi}{2} \left[(\tan \pi - \sec \pi) - (\tan 0 - \sec 0) \right]$$

$$= \frac{\pi}{2} \left[0 - (-1) - 0 + 1 \right] = \frac{\pi}{2} \times 2 = \pi$$

$$\therefore \int_0^{\pi} \frac{x}{1+\sin x} dx = \pi.$$

26. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$.

Sol. $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

Put $x=\tan\theta \Rightarrow dx=\sec^2\theta\,d\theta \Rightarrow 1+x^2=1+\tan^2\theta=\sec^2\theta$

Upper Limit : $x=1 \Rightarrow \tan\theta=1 \Rightarrow \theta=\frac{\pi}{4}$, Lower Limit : $x=0 \Rightarrow \tan\theta=0 \Rightarrow \theta=0$

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

Let $\mathrm{I} = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$

We know that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore\ \mathrm{I} = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4} \log\left[\frac{(1+\tan\theta)+(1-\tan\theta)}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} [\log 2-\log(1+\tan\theta)]\,d\theta$$

$$= \int_0^{\pi/4} \log 2\,d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

$$I = \log 2\int_0^{\pi/4} 1.d\theta - I$$

$$\Rightarrow I + I = \log 2.(\theta)_0^{\pi/4} = \log 2\left(\frac{\pi}{4}\right) - 0$$

$$\Rightarrow 2I = \frac{\pi}{4}\log 2$$

$$\Rightarrow I = \frac{\pi}{8}\log 2$$

27. Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$.

Sol. We know that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Let $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$$

$$\therefore I = \int_0^{\pi} \frac{\pi\sin x - x\sin x}{1+\cos^2 x}dx = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

$$I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx - I$$

$$\therefore I + I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Put $\cos x = t \Rightarrow \sin x dx = -dt$

Upper Limit : $x = \pi \Rightarrow \mathrm{t} = \cos\pi = -1$, Lower Limit : $x = 0 \Rightarrow t = \cos 0 = 1$

$$\Rightarrow 2I = \pi\int_1^{-1} \frac{-dt}{1+t^2}$$

$$\Rightarrow I = -\frac{\pi}{2}\int_1^{-1} \frac{1}{1+t^2}dt$$

$$= -\frac{\pi}{2}[\tan^{-1} t]_1^{-1} = -\frac{\pi}{2}[\tan^{-1}(-1) - \tan^{-1}(1)]$$

$$= -\frac{\pi}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi}{2}\times\frac{\pi}{2} = \frac{\pi^2}{4}$$

28. Evaluate $\int_0^{\pi/4} \log(1+\tan x)\,dx$.

Sol. Let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$

We know that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$\therefore I = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \left(\log 2-\log(1+\tan x)\right)dx = \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$$

$$\therefore I = \int_0^{\pi/4} \log 2\,dx - I$$

$$\Rightarrow I + I = (\log 2)\int_0^{\pi/4} 1.dx$$

$$\Rightarrow 2I = (\log 2)[x]_0^{\pi/4}$$

$$2\,I = \log 2\left(\frac{\pi}{4}-0\right) = \frac{\pi}{4}\log 2$$

$$\therefore I = \frac{\pi}{4\times 2}\log 2$$

$$\Rightarrow \int_0^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2$$

Unit **8**

Differential Equations

Definition: An equation involving one dependent variable and its derivatives with respect to one independent variable is called as Ordinary Differential Equation.

Ex: $\frac{dy}{dx}+5x=\cos x$

$$\left(\frac{d^2y}{dx^2}\right)^2-3\left(\frac{dy}{dx}\right)^3-e^x=4$$

Definition: If a D.E. contains one dependent variable and more than one independent variables, then it is called as Partial D.E.

Ex: $x.\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$, $\qquad \frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}+\frac{\partial^2\omega}{\partial z^2}=0$

where, $z=f(x,y)$ $\qquad$ where, $\omega=f(x,y,z)$

We learn about Ordinary D.E.

Definition: The order of a D.E is the order of the highest order derivative occuring in it.

Definition: The degree of a D.E is the largest exponent of the highest order derivative occuring in it after the equation has been expressed in a form of a polynomial equation in derivatives.

(The exponent of x and y need not be an integer)

1. $\frac{dy}{dx}=\frac{x^{1/2}}{y^{1/2}\left(1+x^{1/2}\right)}$

   order = 1, degree = 1

2. $\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/3}$

$$\Rightarrow\left(\frac{d^2y}{dx^2}\right)^3=\left[1+\left(\frac{dy}{dx}\right)^2\right]^5$$

order = 2, degree = 3

3. $$1+\left(\frac{d^2y}{dx^2}\right)^2=\left[2+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$$\Rightarrow\left[1+\left(\frac{d^2y}{dx^2}\right)^2\right]^2=\left[2+\left(\frac{dy}{dx}\right)^2\right]^3$$

order = 2, degree = 4

4. $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\log\left(\frac{dy}{dx}\right)$$

Order is 2 and Degree is not defined since the equation cannot be expressed as a polynomial equation in the derivatives.

5. $$\frac{d^2y}{dx^2}=-p^2y$$

order = 2, degree = 1

6. $$\left(\frac{d^3y}{dx^3}\right)^2-3\left(\frac{dy}{dx}\right)^2-e^x=4$$

order = 3, degree = 2

7.* $$\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\right]^{6/5}=6y$$

$$\Rightarrow\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=(6y)^{\frac{5}{6}}$$

order = 2, degree = 1

❖ The general form of an ordinary differential equation of n^{th} order is

$$F\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\ldots\ldots,\frac{d^ny}{dx^n}\right)=0$$

Solution of a D.E: A solution of a D.E is a relation between dependent variable, independent variables and along with some arbitrary constants satisfying the D.E.

General Solution : A solution of a D.E in which the number of arbitrary constants is equal to the order of the D.E is called the general solution.

Particular Solution : A particular solution of a D.E is a solution obtained by giving particular values to the arbitrary constants in the general solution.

Very Short Answer Type Questions:

1. Form the D.E corresponding to $y = cx - 2c^2$, where c is a parameter.

sol: Given: $y = cx - 2c^2$ ....(1)

It has only one arbitrary constant

So differentiating once with respect to x, we get

$$\frac{dy}{dx} = c(1) - 0$$

Substituting $c = \frac{dy}{dx}$ in equation (1), 'c' gets eliminated

$\therefore$ The required D.E is $y = \left(\frac{dy}{dx}\right)x - 2\left(\frac{dy}{dx}\right)^2$

2. Form a D.E corresponding to $y = A\cos 3x + B\sin 3x$ where A,B are parameters.

Sol: Given: $y = A\cos 3x + B\sin 3x$ .... (1)

Since there are two arbitrary constants or parameters,

differentiating two times successively with respect to x, we get

$$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -9A\cos 3x - 9B\sin 3x$$

$$= -9\left(A\cos 3x + B\sin 3x\right)$$

$$= -9y \qquad [\because \text{ from } (1)]$$

$\therefore \frac{d^2y}{dx^2} = -9y$ is the required D.E, where A and B are eleminated

$$\frac{d^2y}{dx^2} + 9y = 0$$

3. Find the order of the D.E. obtained by eliminating the arbitrary constants b and c from the equation $xy = ce^x + be^{-x} + x^2$

Sol: There are two arbitrary constants b and c in the equation

$xy = ce^x + be^{-x} + x^2$ ...... (1)

So differentiating twice successively w.r.t. x, we get

$$x\,.\,\frac{dy}{dx}+y.1=ce^{x}+be^{-x}(-1)+2x$$

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}.1+\frac{dy}{dx}=ce^{x}+be^{-x}+2$$

$$\Rightarrow x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dy}{dx}=\left(ce^{x}+be^{-x}\right)+2$$

$$=(xy-x^2)+2 \qquad [\because \text{ from } (1)]$$

$$\Rightarrow x.\frac{d^2y}{dx^2}+2\frac{dy}{dx}=xy-x^2+2 \text{ is the D.E.}$$

$\therefore$ Order = 2.

4. Find the order of the D.E. of the family of all circles with their centres at the origin.

Sol.: The general equation of the circle with centre (0, 0) is

$x^2+y^2=r^2$ .... (1)

r^2 is the arbitrary constant.

So, differentiating equation (1) only once, we get

$$2x+2y\frac{dy}{dx}=0\Rightarrow x+y\frac{dy}{dx}=0$$

$\therefore$ Order = 1.

5. Form the D.E. of the following family of curves where parameters are given in brackets.

(i) $y=c(x-c)^2$ .... (1)

differentiating once w.r.t. x we get

$$\frac{dy}{dx}=c.2(x-c) \text{ .... (2)}$$

Now $\frac{(1)}{(2)}\Rightarrow\frac{y}{\left(\frac{dy}{dx}\right)}=\frac{c(x-c)^2}{c.2(x-c)}$

$$\Rightarrow\frac{y}{\frac{dy}{dx}}=\frac{x-c}{2}$$

$$\Rightarrow\frac{2y}{\frac{dy}{dx}}=x-c$$

$$\Rightarrow c = x - \frac{2y}{\left(\frac{dy}{dx}\right)}$$

Substituting C value in (1) we get,

$$y = \left(x - \frac{2y}{\left(\frac{dy}{dx}\right)} \right) \times \left(\frac{2y}{\left(\frac{dy}{dx}\right)} \right)^2$$

$$y = \frac{x.\frac{dy}{dx} - 2y}{\frac{dy}{dx}} \times \frac{4.y^2}{\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow y\left(\frac{dy}{dx}\right)^3 = \left(x\frac{dy}{dx} - 2y\right)4y^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^3 = 4xy\frac{dy}{dx} - 8y^2$$

(ii) $xy = ae^x + be^{-x}$, *a, b* are parameters

Sol. $xy = ae^x + be^{-x}$ ....(1)

Since there are two parameters, differentiating equation (1) twice sucessively w.r.t.x, we get

$$x.\frac{dy}{dx} + y.1 = ae^x + be^{-x}$$

$$\Rightarrow x.\frac{dy}{dx} + y = ae^x - be^{-x} \qquad ....(2)$$

Again differentiating w.r.t. 'x', we get

$$x.\frac{d^2y}{dx^2} + \frac{dy}{dx}.1 + \frac{dy}{dx} = ae^x + be^{-x} = xy \qquad [\because \text{from (1)}]$$

$x.\frac{d^2y}{dx^2} + 2.\frac{dy}{dx} - xy = 0$ is the required differentiating equation

(iii) $y = a\cos(nx+b)$, *a, b* are parameters

Sol: $y = a\cos(nx+b)$

Since there are two parameters, differentiating (1), twice sucessively w.r.t.x, we get

$$\frac{dy}{dx} = -a\sin(nx+b) \times n \qquad ....(1)$$

$$\Rightarrow \frac{dy}{dx} = -an\sin(nx+b)$$

Again differentiating w.r.t.x.

$$\frac{d^2y}{dx^2} = -a\,n\cos(nx+b)\times n$$

$$= -a\,n^2\cos(nx+b)$$

$$= -n^2\left[a\cos(nx+b)\right]$$

$$= -n^2 y \qquad [\because \text{ from (1)}]$$

$$\Rightarrow \frac{d^2y}{dx^2} = -n^2 y$$ is the required differential equation

Solving Differential Equations:

Methods to solve first order, first degree D.E.

The general first order, first degree D.E. contains the terms of $\frac{dy}{dx}$, x and y.

So it is of the form, $\frac{dy}{dx} = \mathrm{F}(x, y)$ where F is a function of x and y.

Variables Separable Method:

If the given D.E. can be written in the form of $f(x).dx + g(y).dy = 0$, then its solution can be obtained by integrating each term. This method of solving the D.E. is called variables separable method.

Long Answer Type Questions

1. Solve : $x + y\frac{dy}{dx} = 0$

Sol. Given D.E is, $x + y\frac{dy}{dx} = 0$

$$\Rightarrow y\frac{dy}{dx} = -x$$

$$\Rightarrow ydy = -xdx$$

Integrating on both sides,we get

$$\int ydy = \int -xdx$$

$$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = c$$

$\Rightarrow x^2 + y^2 = 2c$, is the required solution

Note: After integration on both sides, write the constant of integration, C, on any one side.

2. Solve $\frac{dy}{dx}=e^{x+y}$

Sol. Given D.E: $\frac{dy}{dx}=e^{x+y}$

$$\Rightarrow \frac{dy}{dx}=e^{x}.e^{y}$$

$$\Rightarrow \frac{dy}{e^{y}}=e^{x}.dx$$

$$\Rightarrow e^{-y}dy=e^{x}dx$$

Integrating on both sides, we get

$$\int e^{-y}dy=\int e^{x}dx$$

$$\Rightarrow -e^{-y}=e^{x}+c$$

$\Rightarrow e^{x}+e^{-y}+c=0$ is the required solution

3. Solve $\frac{dy}{dx}=\frac{y^2+2y}{x-1}$

Sol. Given D.E is $\frac{dy}{dx}=\frac{y^2+2y}{x-1}$

$$\Rightarrow \frac{dy}{y^2+2y}=\frac{dx}{x-1}$$

$$\Rightarrow \int\frac{dy}{y^2+2y}=\int\frac{1}{x-1}dx$$

$$\Rightarrow \int\frac{1}{y^2+2y+1^2-1^2}dy=\log|x-1|+c$$

$$\Rightarrow \int\frac{1}{(y+1)^2-1^2}dy=\log|x-1|+c$$

$$\Rightarrow \frac{1}{2(1)}\log\left|\frac{y+1-1}{y+1+1}\right|=\log|x-1|+\log c$$

$$\Rightarrow \log\left|\frac{y}{y+2}\right|=2\log\left|(x-1)\times c\right|$$

$$\Rightarrow \log\frac{y}{y+2}=\log\left((x-1)\times c\right)^2$$

$$\Rightarrow \log\frac{y}{y+2}=\log(x-1)^2\times c^2$$

$$\Rightarrow \frac{y}{y+2} = c^2 (x-1)^2$$

$\Rightarrow y = c^2 (y+2)(x-1)^2$, is the required solution.

4. Solve $y(1+x)dx + x(1+y)dy = 0$

Sol. $y(1+x)dx + x(1+y)dy = 0$

$$\Rightarrow y(1+x)dx = -x(1+y)dy$$

$$\Rightarrow \frac{y(1+x)}{-x(1+y)} = \frac{dy}{dx}$$

$$\Rightarrow \frac{(1+x)}{-x} \times \frac{y}{1+y} = \frac{dy}{dx}$$

$$\Rightarrow \frac{(1+x)}{-x} dx = \frac{1+y}{y} dy$$

Integrating on both sides, we get

$$\Rightarrow -\int \frac{(1+x)}{x} dx = \int \frac{1+y}{y} dy$$

$$\Rightarrow -\int \left(\frac{1}{x} + \frac{x}{x} \right) dx = \int \left(\frac{1}{y} + \frac{y}{y} \right) dy$$

$$\Rightarrow -\int \left(\frac{1}{x} + 1 \right) dx = \int \left(\frac{1}{y} + 1 \right) dy$$

$$\Rightarrow -[\log x + x] = [\log y + y] + c$$

$$\Rightarrow -\log x - x = \log y + y + c$$

$\Rightarrow x + y + \log x + \log y + c = 0$, is the required solution

5. Solve $\sqrt{1+x^2}\ \sqrt{1+y^2}\ dx + xy\ dy = 0$

Sol. $\Rightarrow xydy = -\sqrt{1+x^2}\ \sqrt{1+y^2}\ dx$

$$\Rightarrow \frac{ydy}{\sqrt{1+y^2}} = -\frac{\sqrt{1+x^2}\,dx}{x}$$

Integrating on both sides, we get

$$\Rightarrow \int \frac{ydy}{\sqrt{1+y^2}} = -\int \frac{\sqrt{1+x^2}\,dx}{x} \qquad -(1)$$

L.H.S :

$$\int \frac{ydy}{\sqrt{1+y^2}}$$

$$=\frac{1}{2}\int \frac{2y}{\sqrt{1+y^2}}dy \qquad \left[\because \int \frac{f^1(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}\right]$$

$$=\frac{1}{2}.2\sqrt{1+y^2}$$

$$\int \frac{ydy}{\sqrt{1+y^2}} = \sqrt{1+y^2}$$

$$\text{R.H.S} = \int \frac{\sqrt{1+x^2}}{x}dx$$

$$=\int \sqrt{1+x^2}.\frac{dx}{x}$$

$$=\int t.\frac{tdt}{t^2-1}$$

$$=\int \frac{t^2}{t^2-1}dt$$

$$=\int \frac{t^2-1+1}{t^2-1}dt$$

$$=\int\left(\frac{t^2-1}{t^2-1}+\frac{1}{t^2-1}\right)dt$$

$$=\int\left(1+\frac{1}{t^2-1}\right)dt$$

$$=t+\frac{1}{2.1}\log\left|\frac{t-1}{t+1}\right|$$

$$=\int \frac{\sqrt{1+x^2}}{x}dx = +\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|+c$$

[$\because$ Put $1+x^2=t^2 \Rightarrow t=\sqrt{1+x^2}$

$$\Rightarrow x^2=t^2-1 \Rightarrow x=\sqrt{t^2-1}$$

$$\Rightarrow 2xdx = 2tdt \quad \Rightarrow dx=\frac{tdt}{x}$$

$$\Rightarrow \frac{dx}{x}=\frac{tdt}{x.x}=\frac{tdt}{x^2}$$

$$\frac{dx}{x}=\frac{tdt}{t^2-1}$$]

Substituting LHS, RHS in (1), we get

$$\sqrt{1+y^2} = -\left[\sqrt{1+x^2}+\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|\right]+c$$

$$\Rightarrow \sqrt{1+y^2}+\sqrt{1+x^2}+\frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|=c$$

$$\Rightarrow \sqrt{1+y^2}+\sqrt{1+x^2}+\frac{1}{2}\log\left(\frac{x}{\sqrt{1+x^2}+1}\right)^2=c$$

$$\left[\because \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\times\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1}=\frac{(1+x^2-1)}{\left(\sqrt{1+x^2}+1\right)^2} =\frac{x^2}{\left(\sqrt{1+x^2}+1\right)^2}=\left(\frac{x}{\sqrt{1+x^2}+1}\right)^2\right]$$

$$\Rightarrow \sqrt{1+y^2}+\sqrt{1+x^2}+\log\left(\frac{x}{\sqrt{1+x^2}+1}\right)=c$$

$$\Rightarrow \sqrt{1+y^2}+\sqrt{1+x^2}+\log x-\log\left(\sqrt{1+x^2}+1\right)=c$$

6. Solve $\sqrt{1-x^2}\,dy+\sqrt{1-y^2}\,dx=0$

Sol. Given $\sqrt{1-x^2}\,dy+\sqrt{1-y^2}\,dx=0$

$$\Rightarrow \sqrt{1-x^2}\,dy=-\sqrt{1-y^2}\,dx$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=\frac{-dx}{\sqrt{1-x^2}}$$

Integrating on both sides, we get

$$\int\frac{1}{\sqrt{1-y^2}}dy=-\int\frac{1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow \sin^{-1}y=-\sin^{-1}x+c$$

$\Rightarrow \sin^{-1}x+\sin^{-1}y=c$ is the required solution.

7. Solve $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$

Sol. $$\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$$

$$\Rightarrow \frac{dy}{1+y^2}=\frac{dx}{1+x^2}$$

Integrating on both sides, we get

$$\Rightarrow \int\frac{dy}{1+y^2}=\int\frac{dx}{1+x^2}$$

$Tan^{-1}y=Tan^{-1}x+c$, is the required solution.

8. Solve $\frac{dy}{dx} = e^{y-x}$

Sol. $\frac{dy}{dx} = e^{y-x}$

$\Rightarrow \frac{dy}{dx} = \frac{e^y}{e^x}$

$\Rightarrow \frac{dy}{e^y} = \frac{dx}{e^x}$

$\Rightarrow e^{-y} dy = e^{-x} dx$

Integrating on both sides, we get

$\Rightarrow \int e^{-y} dy = \int e^{-x} dx$

$\Rightarrow \frac{e^{-y}}{-1} = \frac{e^{-x}}{-1} + c$

$\Rightarrow e^{-y} = e^{-x} + c$, is the solution.

9. Solve $\left(e^x + 1\right) y dy + \left(y+1\right) dx = 0$

Sol. Given $\left(e^x + 1\right) y dy + \left(y+1\right) dx = 0$

$\Rightarrow \left(e^x + 1\right) y dy = -\left(y+1\right) dx$

$\Rightarrow \frac{y dy}{y+1} = \frac{-dx}{e^x + 1}$

Integrating on both sides, we will get

$\Rightarrow \int \frac{y dy}{y+1} = \int \frac{-dx}{e^x + 1}$ ....(1)

L.H.S: $\int \frac{y}{y+1} dy$

$= \int \frac{y+1-1}{y+1} dy$

$= \int \left(\frac{y+1}{y+1} - \frac{1}{y+1} \right) dy = \int \left(1 - \left(\frac{1}{y+1} \right) \right) dy = \int 1 . dy - \int \left(\frac{1}{y+1} \right) dy$

$= y - \log | y+1 |$

RHS: $\int \frac{dx}{e^x+1}$ Put $e^x = t \Rightarrow e^x dx = dt \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t}$.

$= \int \frac{dt}{t(t+1)}$

$= \int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$

$= \log|t| - \log|t+1|$

$\int \frac{dx}{e^x+1} = \log|e^x| - \log|e^x+1| + \log c$

substituting LHS, RHS in (1) we get the required solution as

$y - \log|y+1| = -\log|e^x| + \log|e^x+1| + \log c$

$\Rightarrow y = \log(y+1) - \log e^x + \log(e^x+1) + \log c$

$y = \log_e\left[\frac{(y+1)(e^x+1)c}{e^x}\right]$ $\left(\because \frac{e^x+1}{e^x} = \frac{e^x}{e^x} + \frac{1}{e^x} = 1 + e^{-x}\right)$

$\Rightarrow e^y = \frac{(y+1)(e^x+1).c}{e^x}$

$\Rightarrow e^y = c(y+1)(1+e^{-x})$

Problems for Practice:

1. Solve: $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ Ans: $e^y = e^x + \frac{x^3}{3} + c$

2. Solve: $\tan y\ dx + \tan x\ dy = 0$ Ans: $\sin x . \sin y = c$

10. Solve $\sqrt{1+x^2}\,dx + \sqrt{1+y^2}\,dy = 0$

Sol. $\Rightarrow \sqrt{1+x^2}\,dx = -\sqrt{1+y^2}\,dy$

Integrating on both sides, we get

$\int \sqrt{1+x^2}\,dx = -\int \sqrt{1+y^2}\,dy$

$\Rightarrow \frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\sinh^{-1}(x) = -\left[\frac{y}{2}\sqrt{1+y^2} + \frac{1}{2}\sinh^{-1} y\right] + c$

$\Rightarrow x\sqrt{1+x^2} + y\sqrt{1+y^2} + \sinh^{-1} x + \sinh^{-1} y = 2c$

11. Solve $\frac{dy}{dx}=\frac{xy+y}{xy+x}$

Sol. $\frac{dy}{dx}=\frac{xy+y}{xy+x}$

$\Rightarrow \frac{dy}{dx}=\frac{y(x+1)}{x(y+1)}$

$\Rightarrow \frac{dy}{dx}=\frac{y}{y+1}.\frac{x+1}{x}$

$\Rightarrow \frac{y+1}{y}dy=\frac{x+1}{x}dx$

$\Rightarrow \left(\frac{y}{y}+\frac{1}{y}\right)dy=\left(\frac{x}{x}+\frac{1}{x}\right)dx$

$\Rightarrow \left(1+\frac{1}{y}\right)dy=\left(1+\frac{1}{x}\right)dx$

Integrating on both sides

$\Rightarrow \int\left(1+\frac{1}{y}\right)dy=\int\left(1+\frac{1}{x}\right)dx$

$\Rightarrow y+\log|y|=x+\log|x|+c$, is the required solution.

12. Solve D.E. is $\frac{dy}{dx}=\sqrt{y-x}$ ____(1)

Sol. Put $y-x=t^2$

differentiating w.r.t x

$\frac{dy}{dx}-1=2t\frac{dt}{dx}$

$\Rightarrow \frac{dy}{dx}=1+2t\frac{dt}{dx}$

Subsituting in (1), we get

$1+2t\frac{dt}{dx}=t$

$\Rightarrow \frac{dt}{dx}=\frac{t-1}{2t}$

$\Rightarrow dt=\left(\frac{t-1}{2t}\right)dx \quad \Rightarrow \frac{2tdt}{t-1}=dx$

Integrating on both sides, we get

$$2\int \frac{t.dt}{t-1} = \int dx$$

$$2\int \frac{t-1+1}{t-1}\, dt = x$$

$$2\int \left(\frac{t-1}{t-1} + \frac{1}{t-1}\right) dt = x$$

$$2\int \left(1 - \frac{1}{t-1}\right) dt = x$$

$$2\left[t + \log|t-1|\right] = x + c$$

$$\Rightarrow 2\left[\sqrt{y-x} + \log\sqrt{y-x} - 1\right] = x + c \qquad \left[\because t^2 = y - x \Rightarrow t = \sqrt{y-x}\right]$$

is the required solution of the given D.E.

Problems for Practice

1. Solve: $\frac{dy}{dx} + 1 = e^{x+y}$ Ans: $e^{-(x+y)} + x + c = 0$

[Hint: put $x + y = t$]

13. Solve $\frac{dy}{dx} = (3x + y + 4)^2$

Sol. Given D.E. is $\frac{dy}{dx} = (3x + y + 4)^2$ ___ (1)

put $3x + y + 4 = t$

differentiating w.r.t 'x', we get

$$3.1 + \frac{dy}{dx} + 0 = \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 3$$

Subsituting in (1), we get

$$\frac{dt}{dx} - 3 = t^2$$

$$\Rightarrow \frac{dt}{dx} = t^2 + 3$$

$$\Rightarrow dt = (t^2 + 3)dx.$$

$$\Rightarrow \frac{dt}{t^2+3} = dx.$$

Integrating on both sides, we get

$$\int \frac{dt}{t^2+3} = \int dx$$

$$\Rightarrow \int \frac{1}{t^2+(\sqrt{3})^2}\, dt = \int dx$$

$$\Rightarrow \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) = x + c$$

$\Rightarrow \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right) = x + c$, is the required solution of the given D.E.

14. Solve $\frac{dy}{dx} - x\tan(y-x) = 1$

Sol: Put $y - x = t$ so that $\frac{dy}{dx} - 1 = \frac{dt}{dx}$.

Therefore, the given equation becomes

$$1 + \frac{dt}{dx} - x\tan t = 1$$

(or) $\frac{dt}{dx} = x\tan t.$

Therefoe, $\cot\ t\ dt = x.dx$ so that $\int \cot t\, dt = \int x\ dx$.

Hence, $\log|\sin t| = \frac{x^2}{2} + c$

i.e. $\log|\sin(y-x)| = \frac{x^2}{2} + c$, which is the required solution.

15. Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Sol. Given D.E. is $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

$$\Rightarrow \frac{dy}{dx} = \sin(x+y) \qquad \_\_(1)$$

Put $x + y = t$

differentiating w.r.t to 'x'

$$1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{dt}{dx}-1$$

Substituting in (1), we get

$$\frac{dt}{dx}-1=\sin t$$

$$\Rightarrow \frac{dt}{dx}=1+\sin t$$

$$\Rightarrow dt=(1+\sin t)dx$$

$$\Rightarrow \frac{dt}{1+\sin t}=dx$$

Integrating on both sides, we get

$$\int \frac{dt}{1+\sin t}=\int dx$$

$$\Rightarrow \int\left(\frac{1}{1+\sin t}\times\frac{1-\sin t}{1-\sin t}\right)dt=\int dx$$

$$\Rightarrow \int \frac{1-\sin t}{1-\sin^2 t}\,dt=\int dx$$

$$\Rightarrow \int \frac{1-\sin t}{\cos^2 t}\,dt=\int dx$$

$$\Rightarrow \int\left(\frac{1}{\cos^2 t}-\frac{\sin t}{\cos^2 t}\right)dt=\int dx$$

$$\Rightarrow \int\left(\sec^2 t-\tan t\sec t\right)dt=\int dx$$

$$\Rightarrow \int \sec^2 t.dt-\int \tan t\sec t\,dt=\int dx$$

$$\Rightarrow \tan t-\sec t=x+c$$

$\Rightarrow \tan(x+y)-\sec(x+y)=x+c$, is the required solution of the given D.E.

16. Solve : $\dfrac{dy}{dx}=\tan^2(x+y)$

Sol. Given D.E. is $\dfrac{dy}{dx}=\tan^2(x+y)$ ____ (1)

Put $x+y=t$

differentiating w.r.t to 'x', we get

$$1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{dt}{dx}-1$$

Substituting in (1), we get

$$\frac{dt}{dx}-1=\tan^2 t$$

$$\Rightarrow \frac{dt}{dx}=1+\tan^2 t$$

$$\Rightarrow \frac{dt}{dx}=\sec^2 t$$

$$\Rightarrow dt=\sec^2 t\,.dx$$

$$\Rightarrow \frac{dt}{\sec^2 t}=dx$$

$$\Rightarrow \cos^2 t\ dt\ =dx$$

Integrating on both sides, we get

$$\int \cos^2 t\,dt\ =\int dx$$

$$\Rightarrow \int \frac{1+\cos 2t}{2}\ dt=\int dx$$

$$\Rightarrow \frac{1}{2}\int (1+\cos 2t)\ dt\ =\int dx$$

$$\Rightarrow \frac{1}{2}\left[t+\frac{\sin 2t}{2}\right]=x+c$$

$$\Rightarrow \frac{t+\frac{\sin 2t}{2}}{2}=x+c$$

$$\Rightarrow t+\frac{1}{2}\sin 2t=2x+2c,\qquad \text{put } t=x+y$$

$$\Rightarrow x-y-\frac{1}{2}\sin 2\left(x+y\right)+c=0$$