321005

COMBINED COMPETITIVE EXAMINATION (MAIN)

MATHEMATICS

Paper–I

Time : 3 Hours *Full Marks* : 200

Note : *(1) The figures in the right-hand margin indicate full marks for the questions.*

(2) Attempt five questions in all.

(3) Question No. 1 is compulsory.

1. Answer any *ten* questions from the following: 4×10=40

(a) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(1, 0) = (1, -1)$ and $T(0, 1) = (2, 3)$. Find $T(x, y)$ for any $(x, y) \in \mathbb{R}^2$. Also show that T is one-one and onto.

(b) A function $f: \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = x^2 \sin\frac{1}{x}$ if $x \neq 0$, and $f(0) = 0$. Show that $f'(x)$ exist for $x \neq 0$ but is not continuous at $x = 0$.

(c) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(d) Find a so that the points $(a, 0, 3)$ and $(0, -1, 0)$ are equidistant from the plane $2x - 3y + z = 5$.

(e) Find a solution φ of the equation $y'' - 2y' - 3y = 0$ if $\varphi(0) = 0$ and $\varphi'(0) = 1$.

(f) If $r(t) = t^2\mathbf{i} + t\mathbf{j} - t^3\mathbf{k}$ find $\int_1^2 r \times \frac{d^2 r}{dt^2}\, dt$

(g) As a result of leakage, an electrical capacitor discharges at a rate proportional to the charge. If the charge Q has the value Q_0 at the time $t = 0$, find Q as a function of t.

(*h*) Show that the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ satisfies $A^2 - 4A - 5I = 0$ where I is the identity matrix of order 3×3. Also find A^{-1}.

(*i*) Show that the total energy is the sum of classical kinetic energy and rest mass energy.

(*j*) Show that the central attraction is inversely proportional to r^2 if the central orbit is $\frac{l}{r} = 1 + e \cos \theta$ with pole as the centre.

(*k*) Show that the least velocity with which a body can be projected to have a horizontal range R is $\sqrt{gR}$ m/s and the greatest height attained is $\frac{R}{4}$.

(*l*) A particle rests inside a hollow sphere of radius a. If the coefficient of friction is $\frac{1}{\sqrt{3}}$, find the height of the particle from the lowest point.

2. Answer any ***eight*** questions from the following : 5×8=40

(*a*) Show that every square matrix can be expressed as a sum of a symmetric and a skew-symmetric matrix.

(*b*) Show that $\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx, a \geq 0$ is convergent.

(*c*) Solve the equation $xyp^2 + (x^2 + y^2)p + xy = 0$.

(*d*) Show that $a^x > x^a$ if $x > a \geq e$.

(*e*) Find the matrix of the linear transformation $T : \mathrm{R}^3 \to \mathrm{R}^3$ such that $T(x, y, z) = (2x - y + 2z, y + z, 2x - 2y + z)$ w ith respect to the standard basis. Also, find its rank.

(*f*) Show that the covariant derivative of the either of the fundamental tensors is zero.

(*g*) A particle of mass m is acted upon by a force $m\mu \left(x + \frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a show that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

(*h*) A particle is projected on the inside of a smooth vertical circle of radius a from its lowest point with a velocity u. Show that the particle goes right round the circle if $u^2 > 5ag$.

(*i*) Let f be a differentiable function for all values of x, $-\infty < x < \infty$, such that $f(-3) = -3$. $l(3) = 3$ and $|f'(x)| \leq 1$. Show that $f(0) = 0$.

(j) Solve the equation $y'' + 4y = \cos x$.

3. Answer any ***five*** questions from the following : 8×5=40

(a) Let $f : \mathrm{R}^3 \to \mathrm{R}^3$ be a linear transformation such that $f(\alpha_1) = 1, f(\alpha_2) = -1, f(\alpha_3) = 3$ where $\alpha_1 = (1, 0, -1), \alpha_2 = (-1, 1, 1)$ and $\alpha_3 = (1, 1, 0)$. Find the value of $f(a)$ for any element $a = (a, b, c) \in \mathrm{R}^3$.

(b) Find the equation of the plane through the point (2, 5, –8) and perpendicular to each of the planes $2x - 3y + 4z + 1 = 0$ and $4x + y - 2z + 6 = 0$.

(c) Show that the line $y = m(x + a) + \dfrac{a}{m}$ touches the parabola $y^2 = 4a(x + a)$.

(d) If $f = (x + y + 1)\boldsymbol{i} + \boldsymbol{j} + (-x - y)\boldsymbol{k}$, show that f. curl $f = 0$.

(e) Find the stability of equilibrium of a vessel containing a liquid floating in a liquid.

(f) Show that for any value of x, $-\dfrac{1}{2} \le \dfrac{x}{1+x^2} \le \dfrac{1}{2}$.

(g) Show that $\dfrac{\partial g_{ij}}{\partial x^k} - \dfrac{\partial g_{jk}}{\partial x^i} = [jk, i] - [ij, k]$.

4. Answer any ***four*** questions from the following : 10×4=40

(a) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{bmatrix}$

(b) A heavy particle of weight W, attached to a fixed point by a light inextensible string, describes a circle in a vertical plane. The tension in the string has values mW and nW respectively when the particle is at the highest and the lowest point of its path. Show that $n = m + 6$.

(c) Reduce the equation $3x^2 - 6xy - 5y^2 - 6x + 22 - 17 = 0$ to the standard form. Is the conic a central conic? If so, find the centre.

(d) Prove that the minimum value of $\dfrac{(2x-1)(x-8)}{x^2-5x+4}$ is greater than its maximum value.

(e) If $y = \tan^{-1} x$, show that $(1 + x^2)\, y_{n+1} + 2nxy_n + n(n - 1)\, y_n = 0$.

5. Answer any *two* questions from the following : 20×2=40

(a) (i) If $I_n = \int_0^{\frac{\pi}{2}} (a\cos\theta + b\sin\theta)^2 \, d\theta$, where n is a positive integer not less than 2, show that $n I_n = ab(a^{n-2} + b^{n-2}) + (n-1)(a^2 + b^2) I_{n-2}$.

(ii) Show that curl $\left(\frac{a \times r}{r^2}\right) = -\frac{a}{r^3} + \frac{3r}{r^3}(a.r)$

(b) (i) Prove that the lines in which the plane $x + y + z = 0$ cuts the cone $ayz + bzx + cxy = 0$ are at right angles if $a + b + c = 0$.

(ii) Find the directional derivative of $\phi(x, y, z) = xy + yz + zx$ at the point (1, 2, 0) in the direction of $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Find in which direction is the directional derivative maximum? Find its value. Also find the unit normal and tangent plane to the surface $xy + yz + zx = 2$ at the point (1, 2, 0).

(c) If an area is bounded by two concentric semi-circles with their common bounding diameter in the free surface, prove that the depth of the centre of pressure is

$$\frac{3}{16}\pi \frac{(a+b)(a^2+b^2)}{a^2+b^2+ab}$$ where a and b are the radii.

6. Answer any *four* of the following : 10×4=40

(a) Show that the function $\phi_1(x) = e^x$ is a solution of the differential equation

$$xy'' - (x + 1)y' + y = 0.$$

Find a second independent solution of this differential equation.

(b) A rod of small section and of density ρ, has a small portion of metal of weight $\frac{1}{n}$th that of the rod attached to one extremity. Show that the rod will float at any inclination in a liquid of density σ, if $(n+1)^2 \rho = n^2\sigma$.

(c) Let W_1, W_2, W_3 be subspaces of a vector space. If $W_2 \subseteq W_1$, show that $W_1 \cap W_2 + W_3 = W_2 + W_1 \cap W_3$.

(d) Show that the dimension of the vector space of all real symmetric matrices of order $n \times n$ is $\frac{n(n+1)}{2}$.

(e) Show that the equation of the straight line passing through the vector d and equally inclined to three mutually perpendicular vectors a, b, c is $r = d + t\left(\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}\right)$ for $t \in \mathbb{R}$.

7. Answer any ***two*** questions from the following : 20×2=40

(a) Two particles A and B of mass m and one particle C of mass M are kept on the x-axis in the order A, B, C. Particle A is given a velocity $\boldsymbol{vi}$. Consequently there are two collisions, both of which are completely inelastic. If the net energy loss because of these collisions is $\frac{3}{4}$ of the initial energy, show that $M = 2m$.

(b) A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}\,[\lambda - \log(1+\lambda)]$, where V is the terminal velocity of the particle and λV is its initial velocity. Show that the corresponding time is $\frac{V}{g}\log(1+\lambda)$.

(c) If the pressure of air varies as $\left(1+\frac{1}{m}\right)$-th power of the density, show that, neglecting variation of temperature and gravity, the height of the atmosphere would be equal to $(m+1)$ times the height of the homogeneous atmosphere.

8. Answer any ***two*** questions from the following : 20×2=40

(a) Six equal heavy uniform rods of weight w each are freely joined at their extremities. One rod is fixed in a horizontal position and the system lies in a vertical plane. The mid-points of the two upper non-horizontal rods are connected by a string. Show that the tension of the string is $6\omega\cot\theta$ where θ is the inclination of the non-horizontal rods to the horizontal.

(b) Two equal uniform rods are firmly joined at one end so that the angle between them is α and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in a stable equilibrium according as the length of the rod is > or <$4r\operatorname{cosec}\alpha$

9. Answer the following questions : 10+20+10=40

(a) Show that $\lim_{n\to\infty}(\cos mx)^{\frac{n}{x^2}} = e^{\frac{1}{2}m^2 n}$

(b) Let $f(x,y) = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$ when $x \neq 0$ and $y \neq 0$ and $f(x,0) = f(0,y) = f(0,0) = 0$.

Show that $f_{xy} = f_{yx}$ when $x, y \neq 0$ but $f_{xy}\ (0,0) \neq f_{xy} =\ (0,0)$.

(c) Find the center of gravity of the area enclosed by the curves $y = mx$ and $y^2 = 4ax$.

10. Answer the following questions : 20×2=40

(a) A particle of mass is moving in $+ x$ direction with speed and has momentum p and energy E in the frame S. If S' is moving at a speed v in the standard way and p' and E' are the momentum and energy respectively in S', show that $E'^2 - p'^2c^2\ E^2 - p^2c^2$.

(b) A spherical shell formed of two halves in contact along a vertical plane is filled with water. Show that the resultant pressure on either half of the shell is $\frac{1}{4}\sqrt{13}$ of the total weight of the liquid.